BUCKLING AND DEFORMATION OF HEATED CONICAL SHELLS

By

LU-KANG CHANG

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NOMENCLATURE

E = Young's modulus

F = dimensionless strain function

H = height of conical shell

R = mean radius defined in Fig. 2

T = temperature gradient in the middle surface

T_1, T_2 = uniform temperature rise

a_i', a_i'' = coefficients of the deflection functions defined in Eqs. (IV-3) and (IV-15), respectively

b_i', b_i'' = coefficients of the deflection functions defined in Eqs. (IV-30) and (IV-37), respectively

$\hat{a}_i'$, $\hat{a}_i''$ = coefficients of the deflection functions defined in Eqs. (IV-47) and (IV-50), respectively

e_r, e_φ = meridional and circumferential strains in the middle surface

$e_{r\varphi}$ = shear strain in the middle surface

g, k = temperature distribution factor defined in Eqs. (IV-9) and (IV-29), respectively

h = wall-thickness of conical shell

l = length of conical shell

m_i', m_i'' = numbers defined in Eqs. (IV-53)

n = number defined in Eq. (IV-16)

r, φ = surface coordinates

u, v, w = dimensionless meridional circumferential and inward normal displacements

x, r = length index and meridional distance defined in Eq. (II-36) and Fig. 2

x_o = value of x at the large end

α = thermal expansion coefficient

β = semivertex angle

γ_1, γ_2 = numbers defined in Eqs. (IV-26) and (IV-41), respectively

ϵ_r, ϵ_φ = meridional and circumferential strains

$\epsilon_{r\varphi}$ = shear strain

ζ = number defined in Eq. (IV-61)

θ = transformed coordinate defined in Eqs. (II-31)

$\varkappa_r$, $\varkappa_\varphi$ = meridional and circumferential changes of curvature in the middle surface

$\varkappa_{r\varphi}$ = twist of the middle surface

λ = $(1-\nu)/2$

μ, η = number of half waves in meridional and circumferential direction, respectively

ν = Poisson's ratio

τ_1, τ_2 = temperature coefficients defined in Eqs. (IV-9) and (IV-29)

∇^2, ∇^4 = operator defined in Eq. (II-38)

Superscripts

$'$ = prebuckling state

$''$ = additional qualities during buckling

Subscripts

cr = critical values

min = minimum values

max = maximum values

I, II = condition in Cases I and II

, = partial differentiation with respect to the variables following the comma

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Lu-Kang Chang

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The instability of truncated conical shells due to thermal loadings was studied. The basic governing equations were derived by use of the large deflection theory, and Galerkin's method was employed to integrate the equilibrium equation.

Two cases were considered in this investigation. In one case the temperature varies along the generator and the conical shell is restricted completely at both ends, while in the other case the temperature changes in two principal directions and the cone is constrained along the perimeter but is free of resultant longitudinal force at boundaries. The edges are simply supported.

The relation of the critical temperature to the geometric parameters were evaluated. Three parameters were used, namely: the radius-thickness ratio, the radius-height ratio, and the cone's vertex angle. The radius-thickness ratio has the most significant effect on the value of critical temperature. A higher buckling temperature was found by nonlinear analysis when compared to that obtained by linear analysis.

CHAPTER I

INTRODUCTION

Thin-walled shells have many applications as principal elements of structures. One of the main advantages is weight-saving. Among the various types of shells, the conical and cylindrical shells are most frequently used in structures such as space vehicles and reactors. The main advantages of their applications are to reduce the drag and they are easily manufactured. Similar to a slender bar, a shell will deform when a load is applied. An important phase in the design of thin-walled shells is a study of their instability; that is, to determine the limit of the applied loadings, mechanical and thermal, before the shells become unstable. If an axial load is applied to a column and if it is gradually increased, a condition is reached in which the equilibrium state becomes unstable and a small lateral disturbance will produce a deflection which does not disappear when the lateral force is removed. Von Karman and Tsien [1] noted that some structures, especially the shell-like structures, may experience a state of weak stability such that small blows or other disturbances cause them to snap into badly deformed shapes.

Since the early twentieth century, the buckling problem of cylindrical shells, based on the assumption of infinitesimal deformations has been studied by many authors. Among them, Donnell [2] published his well-known Donnell's eighth-order linear equation of

shells. In the derivation of this equation the assumptions are made that the wall thickness is small compared to the radius of the cylinder, that the distortion to transverse shear is neglected, and that the deformation consists of a large number of waves in the circumferential direction. Three equations of equilibrium in the normal, meridional and circumferential directions are combined, and are represented by one equation through the mathematical technique. Unfortunately, the results predicted by the linear classical theory do not agree with the experimental evidence. In order to explain these discrepancies, von Karman and Tsien [3] made a very important advance in the understanding of the problem of buckling of cylinders subjected to axial compression by assuming a diamond-shaped deflection pattern. They stated that the buckling phenomenon of curved shells could be in general only by means of the large deflection theory. They applied this nonlinear theory and the concept of "snap-through" phenomenon to study the postbuckling behavior of the cylindrical shell under axial compression. The work of von Karman and Tsien has been extended and refined by several authors [4,5,6].

In 1950, Donnell and Wan [7] introduced the concept of imperfection in the analysis of von Karman and Tsien. Their work was extended by Loo [8], who, based on the finite-deformation theory together with a consideration of initial imperfection, studied the buckling of cylindrical shells due to torsion. A continuation of work done by Loo was later investigated by Nash [9]. Cylindrical shells subjected to hydrostatic loads have been studied by Kempner and Crouzet-Pascal [10] and Donnell [11].

The problem of pressurized cylindrical shells under axial compression or bending has been studied by Lu and Nash [12] on the basis of nonlinear finite deflection theory. The results of these finite-deflection studies are in good agreement with the experimental evidence. However, recent papers [13,14] report that when more terms are used in the deflection function the minimum buckling load becomes smaller, and there is no way to predict how many terms should be taken for the approximation.

The problem of stability of conical shells subjected to mechanical loadings has been studied by several authors in recent years. The axisymmetrical buckling of circular cones under axial compression was examined by Seide [15]. The buckling loads corresponding to hydrostatic pressure were investigated by Hoff and Singer [16] and Seide [17]. Mushitari and Sachenkov [18] suggested a transformation of coordinates to study the buckling of simply supported cones subjected to axial compression and normal pressure, and the buckling of internally pressurized cones under axial compression was later investigated by Seide [19]. In all of the above-mentioned references, the prebuckling stresses are assumed unchanged during buckling, and the additional middle surface forces depend on normal deflection only.

The large deflection theory is likely to reveal a "snap-through" type of behavior for conical shells. Schnell [20] used the energy method to solve the nonlinear buckling problem under axial compression and internal pressure. The results gave a better agreement with the experiment than those obtained from the linear theory. The study of nonlinear axisymmetric buckling of the conical shell was also examined

by Newman and Reiss [21] and Famili [22] by the use of finite-difference approximation. The upper and lower buckling loads have been precisely discussed in [22], and Famili's results are seen to be between the results of Niordson [23] and Bijlaard [24], in which Niordson's work is considered as upper bound and Bijlaard's work is considered as lower bound.

The buckling problem of cylindrical shells subject to a uniform temperature gradient has been studied by Hoff [25] for simply supported ends and Zuk [26] for clamped boundary conditions. In these works, the cylinder is restrained circumferentially at both ends but is free from constraint in the axial direction. The longitudinal stress is assumed to be zero, and the hoop stress varies in the axial direction and rapidly decreases as the distance from the edge increases. Hoff approximated the hoop stress by an infinite cosine series and solved the problem by the use of Donnell's equation of thin shell, while Zuk presented the hoop stress by a cosine function and solved the governing shell equation by Galerkin's method. This problem was later analyzed by Sunakawa [27] for various conditions of constraint. He used the axisymmetrical condition in the state prior to buckling, and the problem of finding the hoop stress is therefore much simplified.

Similar to [25], the buckling of thin-walled, cylindrical shells under circumferential temperature gradients was examined by Abir and Nardo [28]. The above analysis is performed with the aid of Donnell's equations. The variation of the thermal stress with the circumferential coordinate is represented by a Fourier series and the stresses are assumed constant through the thickness and in the axial direction.

Hoff, Chao and Madsen [29] have also investigated the buckling of a cylindrical shell heated along an axial strip by means of Donnell's linear equation of cylindrical shells. It is assumed in [29] that the cylindrical shell is very long in the axial direction and thus the axial stress caused by heating has constant values over a substantial length of the heated strip as well as across its entire width; and outside the heated strip the thermal stress is assumed to be zero. The problem was solved for the heated as well as the unheated regions. The radial displacements, the slopes, the bending moments, and the transverse shears are taken to be the same at the boundary for two different regions. In a later report [30], the same thermal buckling problem was solved by using the actual temperature data obtained from experiments. This is in contrast with the exact solution of [29] in which use was made of an idealized representation for the temperature variation in the circumferential direction. The results were found to be in close agreement with those of the experiments, provided the heat strip was not very narrow.

The problem of thermal buckling of conical shells heated along a generator was recently studied by Bendavid and Singer [31]. The solution of this problem is obtained by a Rayleigh-Ritz approach in conjunction with displacement functions modified by a shape factor. In this analysis, a hyperbolic axial thermal stress distribution is assumed, the shell is simply supported and is taken long enough so that the edge effect of the thermal stress may be neglected. The radius of the shell is also assumed large compared to the thickness. The hoop stress induced by the thermal load may also be disregarded.

The axial stress is taken in correspondence with the assumed temperature distribution, and the circumferential displacement is chosen to be zero at the edges during heating.

The purpose of the present investigation is to study the buckling and postbuckling behavior of conical shells under presumed temperature distributions. It is well known that finite-deflection analysis on shells of other geometries has been closer to experimental results than classical small-deflection analysis, and it is also noted that during buckling the normal deflection of the shell is no longer infinitesimal. In the present study, therefore, the terms due to large deformation are included in the basic equations.

Two cases are considered in this study. In one case the temperature varies along the generator and the conical shell is restricted completely at both ends, while in the other case temperature changes in two principal directions and the cone is constrained along the circumference but is free of resultant longitudinal force at boundaries. The conical shell to be treated is assumed simply supported and free from other loadings. The shell is considered to be elastic and isotropic. Its material properties are taken to be independent of temperature, and no dynamic effects are considered. The principle of stationary energy is used to find the governing equations of equilibrium subject to thermal loading. Hence, the buckling and minimum temperatures are found from these equations.

The deflection function is first assumed to satisfy the simply supported boundary condition, the stress function is then found from the compatibility equation, which is carried out by the use of the

nonlinear deflection theory. The solution of this problem is obtained by applying the Galerkin method to the equilibrium equation.

Numerical examples of finding the critical temperatures for different geometrical parameters of the conical shells are given in Chapter IV. The deflection-temperature relations after buckling are plotted in Figs. 12-14. The computation was carried out on the IBM 709 Computer and tne general programs were written in Fortran IV language.

The detailed expressions of the symbols used in Chapter IV are explicitly given in Appendices A, B and C. In Appendix C, the expressions are presented in Fortran language. Since the capacity of the IBM 709 Computer was too small for the program shown in Appendix C, the program was therefore divided into three parts during the numerical calculations.

In order to check the magnitude of the buckling temperature, a test was performed by applying heat uniformly over the surface of the conical shell.

CHAPTER II

BASIC RELATIONS AND FUNDAMENTAL EQUATIONS

1. Geometry of Shells

A surface is a locus of points, whose coordinates can be described as a function of two independent parameters x_1 and x_2. In shell theory a special type of curvilinear coordinate system is usually employed. For any point on the surface, the coordinates are of the type $X_1 = X_1(x_1,x_2)$, $X_2 = X_2(x_1,x_2)$ and $X_3 = X_3(x_1,x_2)$, in which (X_1, X_2, X_3) are rectangular coordinates and the parameters x_1 and x_2 are surface coordinates.

The position vector of any point on the surface can be expressed as:

$$\vec{s} = X_1 \vec{i} + X_2 \vec{j} + X_3 \vec{k} \qquad \text{(II-1)}$$

The distance ds between points with the surface coordinates (x_1, x_2) and $(x_1+dx_1, x_2 + dx_2)$ is determined by

$$ds^2 = d\vec{s} \cdot d\vec{s} = A_o^2 (dx_1)^2 + 2B_o^2 (dx_1 dx_2) + C_o^2 (dx_2)^2 \qquad \text{(II-2)}$$

where A_o^2, B_o^2 and C_o^2 are the fundamental magnitudes of the first order. They are

$$A_o^2 = \frac{\partial \vec{s}}{\partial x_1} \cdot \frac{\partial \vec{s}}{\partial x_1}$$

$$B_o^2 = \frac{\partial \vec{s}}{\partial x_1} \cdot \frac{\partial \vec{s}}{\partial x_2} \qquad \text{(II-3)}$$

$$C_o^2 = \frac{\partial \vec{s}}{\partial x_2} \cdot \frac{\partial \vec{s}}{\partial x_2}$$

For orthogonal surface coordinates B_o vanishes, and the magnitudes of $\frac{\partial \vec{s}}{\partial x_1}$ and $\frac{\partial \vec{s}}{\partial x_2}$ are A_o and C_o, respectively.

The unit normal vector of the surface is

$$\vec{n} = \frac{1}{A_o C_o}\left(\frac{\partial \vec{s}}{\partial x_1} \times \frac{\partial \vec{s}}{\partial x_2}\right) \qquad \text{(II-4)}$$

The fundamental magnitudes of second order are defined as:

$$L_o = \frac{\partial^2 \vec{s}}{\partial x_1^2} \cdot \vec{n} = \frac{1}{A_o C_o} \begin{vmatrix} \frac{\partial^2 X_1}{\partial x_1^2} & \frac{\partial^2 X_2}{\partial x_1^2} & \frac{\partial^2 X_3}{\partial x_1^2} \\ \frac{\partial X_1}{\partial x_1} & \frac{\partial X_2}{\partial x_1} & \frac{\partial X_3}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_3}{\partial x_2} \end{vmatrix}$$

$$M_o = \frac{\partial^2 \vec{s}}{\partial x_1 \partial x_2} \cdot \vec{n} = \frac{1}{A_o C_o} \begin{vmatrix} \frac{\partial^2 X_1}{\partial x_1 \partial x_2} & \frac{\partial^2 X_2}{\partial x_1 \partial x_2} & \frac{\partial^2 X_3}{\partial x_1 \partial x_2} \\ \frac{\partial X_1}{\partial x_1} & \frac{\partial X_2}{\partial x_1} & \frac{\partial X_3}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_3}{\partial x_2} \end{vmatrix}$$

$$N_o = \frac{\partial^2 \vec{\delta}}{\partial x_2^2} \cdot \vec{n} = \frac{1}{A_o C_o} \begin{vmatrix} \frac{\partial^2 X_1}{\partial x_2^2} & \frac{\partial^2 X_2}{\partial x_2^2} & \frac{\partial^2 X_3}{\partial x_2^2} \\ \frac{\partial X_1}{\partial x_1} & \frac{\partial X_2}{\partial x_1} & \frac{\partial X_3}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_3}{\partial x_2} \end{vmatrix} \qquad \text{(II-5)}$$

It is shown in differential geometry that the lines of principal radii of curvature coincide with the coordinate lines if, and only if the coordinates are orthogonal and thus $M_o = 0$. In this case, the principal curvatures are

$$\frac{1}{R_1} = -\frac{L_o}{A_o^2} \quad , \qquad \frac{1}{R_2} = -\frac{N_o}{C_o^2} \qquad \text{(II-6)}$$

These equations affix signs to the principal radii of curvature R_1 and R_2. If the lines of principal curvature are coordinate lines, the theorem of Rodrigues is expressed as follows:

$$\frac{\partial \vec{n}}{\partial x_1} = \frac{1}{R_1} \frac{\partial \vec{\delta}}{\partial x_1} \quad , \qquad \frac{\partial \vec{n}}{\partial x_2} = \frac{1}{R_2} \frac{\partial \vec{\delta}}{\partial x_2} \qquad \text{(II-7)}$$

Let $\pm z$ denote the normal distance from the middle surface. Positive z is measured in the sense of positive normal $\vec{n}$ of the middle surface as shown in Fig. 1. To any set of values of (x_1, x_2, z) there corresponds a point in the shell. The position vector of a point on the middle surface is $\vec{R}$ and the position vector of the corresponding

point at distance z from the middle surface is $\vec{R}$. From Fig. 1 we have

$$\vec{R} = \vec{s} + \vec{n}z \tag{II-8}$$

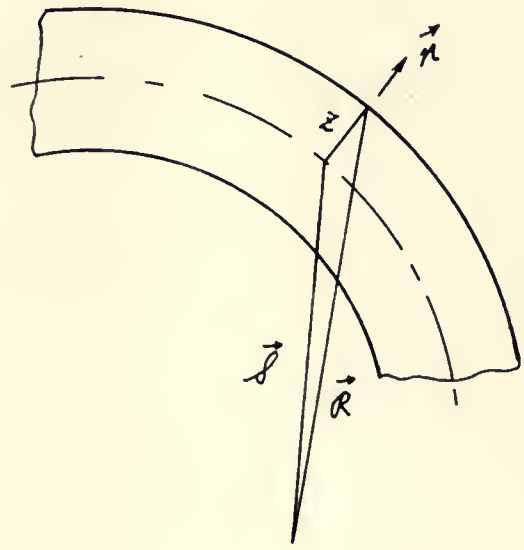


Fig. 1. The cross section of a shell segment.

Hence by differentiation,

$$\frac{\partial \vec{R}}{\partial x_1} = \frac{\partial \vec{s}}{\partial x_1} + z\left(\frac{\partial \vec{n}}{\partial x_1}\right)$$

$$\frac{\partial \vec{R}}{\partial x_2} = \frac{\partial \vec{s}}{\partial x_2} + z\left(\frac{\partial \vec{n}}{\partial x_2}\right) \tag{II-9}$$

Substituting Eq. (II-7) into Eq. (II-9), the following relations are obtained,

$$\frac{\partial \vec{R}}{\partial x_1} = \left(1 + \frac{z}{R_1}\right) \frac{\partial \vec{s}}{\partial x_1}$$

$$\frac{\partial \vec{R}}{\partial x_2} = \left(1 + \frac{z}{R_2}\right) \frac{\partial \vec{s}}{\partial x_2} \qquad \text{(II-10)}$$

$$\frac{\partial \vec{R}}{\partial z} = \vec{n}$$

By referring to a surface parallel to the middle surface at a distance z, we have

$$ds^2 = d\vec{R} \cdot d\vec{R} = \left(\frac{\partial \vec{R}}{\partial x_1} + \frac{\partial \vec{R}}{\partial x_2} + \frac{\partial \vec{R}}{\partial z}\right)^2$$

$$= A^2 (dx_1)^2 + C^2 (dx_2)^2 + \xi^2 (dz)^2 \qquad \text{(II-11)}$$

where

$$A = A_0 \left(1 + \frac{z}{R_1}\right), \quad C = C_0 \left(1 + \frac{z}{R_2}\right), \quad \xi = 1 \qquad \text{(II-12)}$$

The strain-displacement relations are approximated as follows [32]:

$$\epsilon_{11} = \frac{1}{A} \frac{\partial}{\partial x_1} \left[\frac{1}{A_0}\left(A\bar{u} + z\frac{\partial \bar{w}}{\partial x_1}\right)\right] + \frac{1}{AC C_0}\left(\frac{\partial A}{\partial x_1}\right)\left(C\bar{v} + z\frac{\partial \bar{w}}{\partial x_2}\right) - \frac{\bar{w}}{A}\left(\frac{\partial A}{\partial z}\right) - \frac{1}{2A_0^2}\left(\frac{\partial \bar{w}}{\partial x_1}\right)^2$$

$$\epsilon_{22} = \frac{1}{C} \frac{\partial}{\partial x_2} \left[\frac{1}{C_0}\left(C\bar{v} + z\frac{\partial \bar{w}}{\partial x_2}\right)\right] + \frac{1}{AC A_0}\left(\frac{\partial C}{\partial x_1}\right)\left(A\bar{u} + z\frac{\partial \bar{w}}{\partial x_1}\right) - \frac{\bar{w}}{C}\left(\frac{\partial \bar{w}}{\partial z}\right) - \frac{1}{2C_0^2}\left(\frac{\partial \bar{w}}{\partial x_2}\right)^2$$

$$\epsilon_{12} = \frac{C}{A}\frac{\partial}{\partial x_1}\left[\frac{1}{C C_0}\left(C\bar{v} + z\frac{\partial \bar{w}}{\partial x_2}\right)\right] + \frac{A}{C}\frac{\partial}{\partial x_2}\left[\frac{1}{A A_0}\right.$$

$$\left. \times \left(A\bar{u} + z\frac{\partial \bar{w}}{\partial x_1}\right)\right] + \frac{1}{A_0 C_0}\left(\frac{\partial \bar{w}}{\partial x_1}\right)\left(\frac{\partial \bar{w}}{\partial x_2}\right)$$

(II-13)

where $\bar{u}$, $\bar{v}$ and $\bar{w}$ are the displacements of a point in the middle surface in x_1, x_2 and z directions, respectively. $\bar{w}$ is taken inward normal to the middle surface as positive.

Eqs. (II-13) are obtained under the assumption that the transverse shears vanish. In small deflection theories all quadratic terms in the strain-displacement relations are neglected.

Letting e_1, e_2 and e_{12} denote the values of ϵ_1, ϵ_2 and ϵ_{12} on the middle surface, respectively, we find [32]:

$$e_1 = \frac{1}{A_0}\left(\frac{\partial \bar{u}}{\partial x_1}\right) + \frac{\bar{v}}{A_0 C_0}\left(\frac{\partial A}{\partial x_2}\right) - \frac{\bar{w}}{R_1} + \frac{1}{2A_0^2}\left(\frac{\partial \bar{w}}{\partial x_1}\right)^2$$

$$e_2 = \frac{1}{C_0}\left(\frac{\partial \bar{v}}{\partial x_2}\right) + \frac{\bar{u}}{A_0 C_0}\left(\frac{\partial C_0}{\partial x_1}\right) - \frac{\bar{w}}{R_2} + \frac{1}{2C_0^2}\left(\frac{\partial \bar{w}}{\partial x_2}\right)^2$$

$$e_{12} = \frac{1}{A_0}\left(\frac{\partial \bar{v}}{\partial x_1}\right) + \frac{1}{C_0}\left(\frac{\partial \bar{u}}{\partial x_2}\right) - \frac{\bar{u}}{A_0 C_0}\left(\frac{\partial A_0}{\partial x_2}\right) -$$

$$\frac{\bar{v}}{A_0 C_0}\left(\frac{\partial C_0}{\partial x_1}\right) + \frac{1}{A_0 C_0}\left(\frac{\partial \bar{w}}{\partial x_1}\right)\left(\frac{\partial \bar{w}}{\partial x_2}\right)$$

(II-14)

The changes of curvatures $\varkappa_1$, $\varkappa_2$ and $\varkappa_{12}$ on the middle surface have the following forms,

$$\varkappa_1 = -\frac{1}{A_0}\frac{\partial}{\partial x_1}\left[\frac{1}{A_0}\left(\frac{\partial \bar{w}}{\partial x_1}\right)\right] + \frac{1}{A_0 C_0^2}\left(\frac{\partial \bar{w}}{\partial x_2}\right)$$

$$\varkappa_2 = \frac{1}{A_0^2 C_0}\left(\frac{\partial C_0}{\partial x_1}\right)\left(\frac{\partial \bar{w}}{\partial x_1}\right) + \frac{1}{C_0}\frac{\partial}{\partial x_1}\left[\frac{1}{C_0}\left(\frac{\partial \bar{w}}{\partial x_2}\right)\right]$$

$$\varkappa_{12} = \frac{2}{A_0 C_0}\left[\frac{1}{A_0}\left(\frac{\partial A_0}{\partial x_2}\right)\left(\frac{\partial \bar{w}}{\partial x_1}\right) - \frac{1}{B_0}\left(\frac{\partial C_0}{\partial x_1}\right)\left(\frac{\partial \bar{w}}{\partial x_2}\right) - \frac{\partial^2 \bar{w}}{\partial x_1 \partial x_2}\right]$$

(II-15)

2. Basic Equations

A truncated conical shell as shown in Fig. 2, on page 25 is considered. The distance r is measured along the generator from the vertex to a point on the middle surface, and β is the semi-vertex angle. Thus, the surface coordinates are taken as: $x_1 = r$, $x_2 = \varphi$. A point on the surface can be represented parametrically as:

$$X_1 = r \sin\beta \cos\varphi$$

$$X_2 = r \sin\beta \sin\varphi \qquad \text{(II-16)}$$

$$X_3 = r \cos\beta$$

Substituting Eqs. (II-16) into Eqs. (II-3), (II-5) and (II-6), we have

$$A_o = 1 , \qquad C_o = r \sin\beta$$

$$R_1 = \infty , \qquad R_2 = r \tan\beta \tag{II-17}$$

Referring to the above equations, the strain-displacement relations for a conical shell on the middle surface are obtained from Eqs. (II-14),

$$e_r = \bar{u}_{,r} + \frac{1}{2}(\bar{w}_{,r})^2$$

$$e_\varphi = \frac{\bar{u}}{r} - \frac{\bar{w}}{r}\cot\beta + (r\sin\beta)^{-1}\bar{v}_{,\varphi} + \frac{1}{2}[(r\sin\beta)^{-1}\bar{w}_{,\varphi}]^2 \tag{II-18}$$

$$e_{r\varphi} = \bar{v}_{,r} - \frac{\bar{v}}{r} + (r\sin\beta)^{-1}\bar{u}_{,\varphi} + (r\sin\beta)^{-1}\bar{w}_{,r}\,\bar{w}_{,\varphi}$$

and a comma, followed by subscripts indicates differentiation with respect to subscripted variables.

From Eqs. (II-15), the changes of curvature are found

$$\varkappa_r = \bar{w}_{,rr}$$

$$\varkappa_\varphi = \frac{1}{r}\bar{w}_{,r} + (r\sin\beta)^{-2}\bar{w}_{,\varphi\varphi} \tag{II-19}$$

$$\varkappa_{r\varphi} = -2(r\sin\beta)^{-1}\left(\frac{1}{r}\bar{w}_{,\varphi} + \bar{w}_{,r\varphi}\right)$$

The conical shell is assumed to be perfectly elastic and the material properties are independent of the change of temperature; the stress-strain relations are given by the following:

$$\sigma_r = \frac{E}{1-\nu^2}(e_r + \nu e_\varphi) - \frac{\alpha E}{1-\nu} T$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}(e_\varphi + \nu e_r) - \frac{\alpha E}{1-\nu} T \qquad \text{(II-20)}$$

$$\sigma_{r\varphi} = \frac{E}{2(1+\nu)} e_{r\varphi}$$

where E, α, ν, and T represent Young's modulus, thermal expansion coefficient, Poisson's ratio and the temperature difference, respectively.

The equation of equilibrium can be obtained by applying the principle of stationary energy. The strain energy of the shell is written as:

$$U = U_m + U_b + U_T \qquad \text{(II-21)}$$

where U_m and U_b are the membrane and bending strain energy, respectively, while U_T represents the part of the strain energy that results from heating. The expressions of the strain energy are given as

$$U_m = \frac{Eh}{2(1-\nu^2)} \iint_A [e_r^2 + e_\varphi^2 + 2\nu e_r e_\varphi + \tfrac{1}{2}(1-\nu) e_{r\varphi}^2] r \sin\beta \, dr \, d\varphi$$

$$U_b = \frac{\bar{D}}{2} \iint_A [\kappa_r^2 + \kappa_\varphi^2 + 2\nu \kappa_r \kappa_\varphi + \tfrac{1}{2}(1-\nu) \kappa_{r\varphi}^2] r \sin\beta \, dr \, d\varphi$$

$$U_T = -\frac{E}{1-\nu}\iint_A [(e_r + e_\phi)h\bar{T} + (\kappa_r + \kappa_\phi)\bar{M}_T] r\sin\beta\, dr\, d\phi \tag{II-22}$$

where h is wall thickness and

$$\bar{T} = \frac{1}{h}\int_{-h/2}^{h/2} \alpha T\, dz, \quad \bar{M}_T = \int_{-h/2}^{h/2} \alpha T z\, dz, \quad \bar{D} = \frac{E h^3}{12(1-\nu^2)} \tag{II-23}$$

If there is no external force, and equilibrium exists, the principle of stationary energy requires that the strain energy U satisfies the Euler's equations of the calculus variation. Symbolically, this can be expressed as

$$\delta U = 0 \tag{II-24}$$

The functional U has the following form

$$U = \iint_A \bar{\Phi}(r, \phi, \bar{u}, \bar{v}, \bar{w}, \bar{u}_{,r}, \bar{v}_{,r}, \bar{w}_{,r}, \bar{u}_{,\phi}, \bar{v}_{,\phi}, \bar{w}_{,\phi},$$

$$\bar{u}_{,rr}, \bar{v}_{,rr}, \bar{w}_{,rr}, \bar{u}_{,r\phi}, \bar{v}_{,r\phi}, \bar{w}_{,r\phi}, \bar{u}_{,\phi\phi}, \bar{v}_{,\phi\phi}, \bar{w}_{,\phi\phi}) dr\, d\phi \tag{II-25}$$

The Euler equation for $\bar{u}$ is

$$\frac{\partial \bar{\Phi}}{\partial \bar{u}} - \frac{\partial}{\partial r}\left(\frac{\partial \bar{\Phi}}{\partial \bar{u}_{,r}}\right) - \frac{\partial}{\partial \phi}\left(\frac{\partial \bar{\Phi}}{\partial \bar{u}_{,\phi}}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial \bar{\Phi}}{\partial \bar{u}_{,rr}}\right)$$

$$+ \frac{\partial^2}{\partial r \partial \phi}\left(\frac{\partial \bar{\Phi}}{\partial \bar{u}_{,r\phi}}\right) + \frac{\partial^2}{\partial \phi^2}\left(\frac{\partial \bar{\Phi}}{\partial \bar{u}_{,\phi\phi}}\right) = 0 \tag{II-26}$$

the Euler equation for $\bar{v}$ is

$$\frac{\partial \bar{\Phi}}{\partial \bar{v}} - \frac{\partial}{\partial r}\left(\frac{\partial \bar{\Phi}}{\partial \bar{v}_{,r}}\right) - \frac{\partial}{\partial \varphi}\left(\frac{\partial \bar{\Phi}}{\partial \bar{v}_{,\varphi}}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial \bar{\Phi}}{\partial \bar{v}_{,rr}}\right)$$

$$+ \frac{\partial^2}{\partial r \partial \varphi}\left(\frac{\partial \bar{\Phi}}{\partial \bar{v}_{,r\varphi}}\right) + \frac{\partial^2}{\partial \varphi^2}\left(\frac{\partial \bar{\Phi}}{\partial \bar{v}_{,\varphi\varphi}}\right) = 0 \qquad \text{(II-27)}$$

and the Euler equation for $\bar{w}$ is

$$\frac{\partial \bar{\Phi}}{\partial \bar{w}} - \frac{\partial}{\partial r}\left(\frac{\partial \bar{\Phi}}{\partial \bar{w}_{,r}}\right) - \frac{\partial}{\partial \varphi}\left(\frac{\partial \bar{\Phi}}{\partial \bar{w}_{,\varphi}}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial \bar{\Phi}}{\partial \bar{w}_{,rr}}\right)$$

$$+ \frac{\partial^2}{\partial r \partial \varphi}\left(\frac{\partial \bar{\Phi}}{\partial \bar{w}_{,r\varphi}}\right) + \frac{\partial^2}{\partial \varphi^2}\left(\frac{\partial \bar{\Phi}}{\partial \bar{w}_{,\varphi\varphi}}\right) = 0 \qquad \text{(II-28)}$$

The Euler equations for $\bar{u}$ and $\bar{v}$ find the two equilibrium equations in the middle surface, which correspond to the equilibrium state in meridional and circumferential directions, respectively. In order to reduce the number of unknowns, the Airy stress function $\bar{F}$ is defined to satisfy the equations of equilibrium in the $\bar{u}$ and $\bar{v}$ directions

$$\sigma_r = \frac{1}{r}\bar{F}_{,r} + (r\sin\beta)^{-1}\bar{F}_{,\varphi\varphi}$$

$$\sigma_\varphi = \bar{F}_{,rr} \qquad \text{(II-29)}$$

$$\sigma_{r\varphi} = -\left[(r\sin\beta)^{-1}\bar{F}_{,\varphi}\right]_{,r}$$

By substituting Eqs. (II-18) into Eqs. (II-20), the stress-displacement relations are obtained,

$$\sigma_r = \frac{E}{1-\nu^2}\left\{ \bar{u}_{,r} + \frac{1}{2}(\bar{w}_{,r})^2 + \nu\left[\bar{v}_{,r} - \frac{\bar{w}}{r}\cot\beta \right.\right.$$

$$\left.\left. + (r\sin\beta)^{-1}\bar{v}_{,\varphi} + \frac{1}{2}(r\sin\beta)^{-1}\bar{w}_{,\varphi}\right]\right\} - \frac{\alpha E}{1-\nu}T$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}\left\{ \bar{v}_{,r} - \frac{\bar{w}}{r}\cot\beta + (r\sin\beta)^{-1}\bar{v}_{,\varphi} + \frac{1}{2}\left[(\sin\beta)^{-1}\bar{w}_{,\varphi}\right]^2 \right.$$

$$\left. + \nu\left[\bar{u}_{,r} + \frac{1}{2}(\bar{w}_{,r})^2\right]\right\} - \frac{\alpha E}{1-\nu}T$$

$$\sigma_{r\varphi} = \frac{E}{2(1-\nu)}\left[\bar{v}_{,r} - \frac{\bar{v}}{r} + (r\sin\beta)^{-1}\bar{u}_{,\varphi} + (r\sin\beta)^{-1} \times \right.$$

$$\left. \bar{w}_{,r}\bar{w}_{,\varphi}\right] \qquad \text{(II-30)}$$

We introduce the following dimensionless notations,

$$\theta = \varphi\sin\beta, \quad F = \bar{F}/Eh^2, \quad w = \bar{w}/h \qquad \text{(II-31)}$$

After substituting Eqs. (II-29) into Eqs. (II-30) and eliminating the displacement variables $\bar{u}$ and $\bar{v}$, the compatibility equation is found as the following:

$$\bar{\nabla}^4 F = -\left(\frac{\gamma_1^4}{\gamma h}\cot\beta\right)\bar{w}_{,rr} - \frac{\gamma_1^4}{\gamma^2}\left(\bar{w}_{,rr}\bar{w}_{,\theta\theta}\right) + \frac{\gamma_1^4}{\gamma^4}\left(w_{,\theta\theta}\right)^2$$

$$+ \frac{\gamma_1^4}{\gamma^2}\left(w_{,r\theta}\right)^2 - \frac{2\gamma_1^4}{\gamma^3}\left(w_{,r\theta}\, w_{,\theta}\right) - \frac{\gamma_1^4}{\gamma}\left(w_{,r} w_{,rr}\right)$$

$$- \frac{\alpha\gamma_1^3}{h\gamma^2}\left[\gamma_1\left(T_{,r} + \frac{\gamma_1^2}{\gamma} T_{,rr}\right) + T_{,\theta\theta}\right] \qquad \text{(II-32)}$$

where

$$\bar{\nabla}^4 = \bar{\nabla}^2 \cdot \bar{\nabla}^2, \qquad \bar{\nabla}^2 = \gamma_1^2\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] \qquad \text{(II-33)}$$

By substituting Eqs. (II-18), (II-19), (II-20), (II-29), (II-30) and (II-31) into Eq. (II-22) and using relation (II-28), the equation of equilibrium in normal direction is found to be of the following form:

$$D\bar{\nabla}^4 w = \frac{\gamma_1^3}{\gamma}\left(\frac{\gamma_1}{h}\cot\beta\right)F_{,rr} - \frac{1}{1-\nu}\bar{\nabla}^2\bar{M}_T$$

$$+ F_{,rr}\left(\frac{\gamma_1^4}{\gamma} w_{,r} + \frac{\gamma_1^4}{\gamma^2} w_{,\theta\theta}\right) - 2\gamma_1^4\left(\frac{1}{\gamma}F_{,\theta}\right)_{,r} \qquad \text{(II-34)}$$

$$\times\left(\frac{1}{\gamma} w_{,\theta}\right)_{,r} + \gamma_1^4\left(\frac{1}{\gamma}F_{,r} + \frac{1}{\gamma^2}F_{,\theta\theta}\right)w_{,rr}$$

where

$$M_T = \frac{\gamma_1^2}{h^2}\int_{-h/2}^{h/2} \alpha T z\, dz, \qquad D = \frac{1}{12(1-\nu^2)} \qquad \text{(II-35)}$$

3. Transformation of Coordinates

For convenience, the following transformation is introduced

$$x = \log \frac{r}{r_1} \quad , \quad x_o = \log\left(1 + \frac{\ell}{r_1}\right) \tag{II-36}$$

By substituting Eqs. (II-36) into Eq. (II-32), the dimensionless compatibility in the newly defined coordinates is

$$\nabla^4 F = e^{-4x}\Big[-\left(\frac{r_1}{h}\cot\beta\right)e^{x}(w_{,xx} - w_{,x}) - w_{,\theta\theta}(w_{,xx} - w_{,x}) + (w_{,\theta})^2 + (w_{,x\theta})^2 - 2\,w_{,\theta}\,w_{,x\theta} - w_{,x}(w_{,xx} - w_{,x})\Big] - \frac{r_1^2}{h^2}\nabla^2(\alpha T) \tag{II-37}$$

where

$$\nabla^4 = \nabla^2 \cdot \nabla^2, \quad \nabla^2 = e^{-2x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}\right) \tag{II-38}$$

The dimensionless equation of equilibrium after the transformation becomes

$$\nabla^4 w = e^{-4x}\Big[\left(\frac{r_1}{h}\cot\beta\right)e^{x}(F_{,xx} - F_{,x}) + (F_{,xx} - F_{,x})(w_{,x} + w_{,\theta\theta}) + (F_{,\theta\theta} + F_{,x})(w_{,xx} - w_{,x}) - 2(F_{,x\theta} - F_{,\theta})(w_{,x\theta} - w_{,\theta})\Big] \tag{II-39}$$

It is noted in Eq. (II-34) that the thermal moment due to the temperature gradient across the wall-thickness is disregarded because the temperature is assumed uniform throughout the thickness. Eqs. (II-37) and (II-39) are the basic equations for the buckling of conical shells due to the temperature changes.

CHAPTER III

METHOD OF APPROACH

Since the governing equations of the buckling problem have been formulated, our next task is to seek the solutions of the established differential equations together with the boundary conditions. To determine the critical temperature at which the conical shell becomes unstable, the deflection is infinitesimal and the linearized relations in Eqs. (II-37) and (II-39) are used. For the buckling of shells, the two coupled nonlinear equations (II-37) and (II-39) will be solved simultaneously. It is virtually impossible to obtain an exact solution, and only approximate numerical methods will be sought. One of the powerful methods is the method of Galerkin [33,34], which is not only closely related to the variational method but also parallels Ritz's method.

The Galerkin method is briefly explained below. A differential equation can be expressed in the form

$$L(P) = 0 \tag{III-1}$$

if L(P) exists in a two-dimensional domain, an approximate solution of P is assumed in a series form, i.e.,

$$P_n(x,y) = \sum_{i=1}^{n} a_i f_i(x,y) \tag{III-2}$$

where P_n is the approximate solution of P, the f_i's are appropriate functions which satisfy the given boundary conditions, and the a_i's are constant coefficients. By substituting Eq. (III-2) into Eq. (III-1), we find that $L(P_n) = M$, where, in general, $M \neq 0$. To minimize M, Galerkin applied a set of orthogonal conditions,

$$\iint_{\mathscr{D}} L(P_n) f_i(x,y)\, dx\, dy = 0 \qquad \text{(III-3)}$$

$i = 1, 2, 3, \ldots n$

where $\mathscr{D}$ represents the two-dimensional domain and the coefficients a_i can be determined by solving the n algebraic equations in Eq. (III-3) simultaneously.

1. Boundary Conditions

The cone, as shown in Fig. 2, page 25, is considered simply supported and has zero circumferential strain at the edges. The conditions for normal deflection at the two ends are expressed as:

$$w = 0 \qquad \text{at} \qquad x = 0 \quad \text{and} \quad x = x_0 \qquad \text{(III-4)}$$

$$w_{,xx} - (1-\nu) w_{,x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = x_0 \qquad \text{(III-5)}$$

When the edges of the shell are completely restrained in circumferential direction, the following two conditions are obtained:

At $x = 0$

$$F_{,xx} - (1+\nu) F_{,x} - \nu F_{,\theta\theta} + \frac{r_i^2}{h^2} \alpha T = 0 \qquad \text{(III-6)}$$

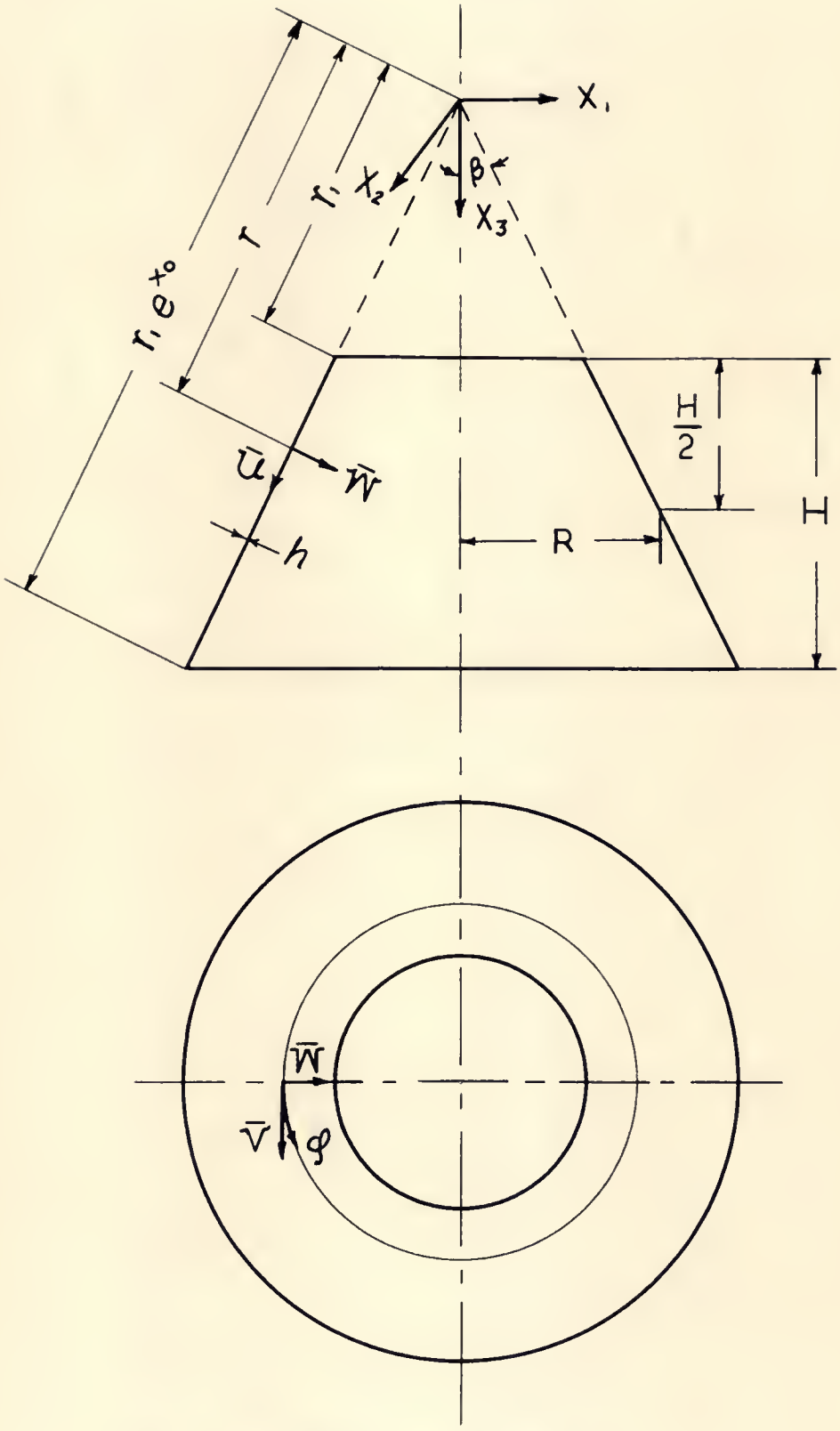


Fig. 2. Geometry of conical shell.

and at $x = x_o$

$$F_{,xx} - (1+\nu) F_{,x} - \nu F_{,\theta\theta} + \frac{\gamma_1^2}{h^2} e^{2x_o} \alpha T = 0 \qquad \text{(III-7)}$$

The end conditions along the direction of the generator will be discussed in the individual cases to be considered.

2. Thermal Buckling

When Eqs. (II-37) and (II-39) are solved simultaneously, the coupled relations are nonlinear. Now we consider only the linear terms in Eqs. (II-37) and (II-39) as well as in the conditions of constraint.

This problem will be approached by two steps: the deflection and the stress function in the prebuckling state are found first; thereafter, the deflection functions and the stress function during buckling are considered. The compatibility equation and equation of equilibrium in the prebuckling state are written, respectively, in the following forms:

$$\nabla^4 F' = -\left(\frac{\gamma_1}{h} \cot\beta\right) e^{-3x} (w'_{,xx} - w'_{,x}) - \frac{\gamma_1^2}{h^2} \alpha \nabla^2 T \qquad \text{(III-8)}$$

and

$$D \nabla^4 w' = \left(\frac{\gamma_1}{h} \cot\beta\right) e^{-3x} (F'_{,xx} - F'_{,x}) \qquad \text{(III-9)}$$

The operators ∇^2 and ∇^4 are defined in Eq. (II-38), and the superscript ($'$) is used for functions in the prebuckling state. The deflection function w', which is assumed in a series form, satisfies the

simply supported boundary conditions given in Eqs. (III-4) and (III-5). The solution of the stress function F', which is the summation of the general solution of the homogeneous equation, and a particular integral, satisfy Eq. (III-8). The arbitrary constants of the general solution of F' are determined in accordance with the conditions of constraint. By applying the Galerkin method to Eq. (III-9), the coefficients of the assumed deflection function w' are then found.

When Eq. (III-8) is substituted into Eq. (II-37), and after eliminating the nonlinear terms, the compatibility equation becomes

$$\nabla^4 F'' = -\left(\frac{\gamma_1}{h}\cot\beta\right) e^{-3x}\left(w''_{,xx} - w''_{,x}\right) \qquad \text{(III-10)}$$

where

$w'' = w - w'$ = additional displacement during buckling

and

$F'' = F - F'$ = additional stress function during buckling.

When relation (III-9) is used in Eq. (II-39), and the terms containing the products of the additional stresses during buckling found from F'' with the derivative of w, are neglected, the equilibrium equation is

$$D\nabla^4 w'' = \left[\left(\frac{r_1}{h}\cot\beta\right)e^{x}\left(F''_{,xx} - F''_{,x}\right)\right.$$

$$+ \left(F'_{,xx} - F'_{,x}\right)\left(w_{,x} + w_{,\theta\theta}\right) + \left(F'_{,x} + F'_{,\theta\theta}\right)$$

$$\left.\times\left(w_{,xx} - w_{,x}\right) - 2\left(F'_{,x\theta} - F'_{,\theta}\right)\left(w_{,x\theta} - w_{,\theta}\right)\right]e^{-4x} \qquad \text{(III-11)}$$

In a similar manner to the determination of prebuckling stress function F', F'' is then found corresponding to the assumed deflection function w'' and the conditions of constraint. After applying the Galerkin method to Eq. (III-11), a set of linear algebraic equations is obtained. The critical temperature can be determined by minimizing the results obtained from these algebraic equations with respect to the wave numbers along generator and circumference.

3. Postbuckling Behavior

The basic equations used for postbuckling problems are Eqs. (II-37) and (II-39). After subtracting Eqs. (III-8) and (III-9) from Eqs. (II-37) and (II-39), respectively, it is found that

$$e^{4x}\nabla^4 F'' = e^{x}\left(-\frac{r_1}{h}\cot\beta\right)\left(w''_{,xx} - w''_{,x}\right) - \left(w_{,\theta\theta}\right.$$

$$\left.+ w_{,x}\right)\left(w_{,xx} - w_{,x}\right) + \left(w_{,x\theta}\right)^2 - 2\left(w_{,x\theta}\right)$$

$$\times\left(w_{,\theta}\right) + \left(w_{,\theta}\right)^2 \qquad \text{(III-12)}$$

$$D\nabla^4 w'' = e^{-4x}\left[\left(\frac{\gamma}{h}\cot\beta\right)e^{x}\left(F,_{xx}'' - F,_{x}''\right) + \left(F,_{xx} - F,_{x}\right)\right.$$

$$\times \left(w,_{x} + w,_{\theta\theta}\right) - 2\left(F,_{x\theta} - F,_{\theta}\right)\left(w,_{x\theta} - w,_{\theta}\right)$$

$$\left. + \left(F,_{x} + F,_{\theta\theta}\right)\left(w,_{xx} - w,_{x}\right)\right] \qquad \text{(III-13)}$$

where

$$F'' = F - F'$$

$$w'' = w - w' \qquad \text{(III-14)}$$

The above equations have the same expression as defined in the previous section; however, the relation between F'' and w'' is different from the linear case.

The deflection function w'', which is assumed in a series form, satisfies the boundary conditions. In a similar manner to the determination of the stress function in the linear case, the stress function during buckling F'' is found to satisfy the compatibility equation and the conditions of constraint. Since Eq. (III-12) is nonlinear, F'' should be a nonlinear function of the coefficients of the deflection function w''.

By substituting the deflection function w and its corresponding stress function F into Eq. (III-13), and integrating it by the Galerkin method, a set of nonlinear algebraic equations is obtained.

The temperature can be determined by solving the algebraic equations, and the minimum temperature is determined by minimizing the temperature T with respect to the number of waves in axial and circumferential directions.

CHAPTER IV

SOLUTION AND NUMERICAL RESULTS

1. Critical Temperature

Two different temperature gradients and their corresponding conditions of constraint are considered in the solution of critical temperature.

Case I

In this case, the conical shell is subjected to a meridional temperature gradient and restricted completely at both ends. Thus, in addition to the boundary conditions expressed in Eqs. (III-4) - (III-7), another boundary condition in meridional direction is

$$\int_0^{x_0} e_r \, dx = \int_0^{x_0} u_{,x} \, dx = 0 \qquad \text{(IV-1)}$$

By substituting Eqs. (II-20) and (II-29) into Eq. (IV-1), the dimensionless condition of constraint expressed in terms of stress function F is obtained as:

$$\int_0^{x_0} e^{-x}\left[(1+\nu)F_{,x} - \nu F_{,xx} + F_{,\theta\theta} + e^{2x}\left(\frac{\pi^2}{h^2}\alpha T\right)\right] dx = 0 \qquad \text{(IV-2)}$$

Since the temperature field considered in this case varies only in the meridional direction, the prebuckling deflection and stress function are therefore taken to be axisymmetric, i.e., they

are independent of θ . The functions with subscript I are correspondent to Case I considered here.

The deflection function in the prebuckling state is assumed to satisfy the simply supported boundary condition, and in accordance with the assumption of Mushitari [18], the prebuckling deflection is chosen in the form:

$$w_I' = e^{\lambda x} \sum_{i=1}^{M} a_i' \sin \frac{i\pi x}{x_0} \tag{IV-3}$$

in which $$\lambda = \frac{1-\nu}{2} \tag{IV-4}$$

In this study, i is taken as an odd integer, M = 5 and ν = 0.3. The coefficients a_i' (i = 1, 3, 5) are in terms of the temperature gradient and will be determined later. According to Eq. (III-8), the prebuckling stress function can be written in the following form:

$$F_I' = A_0' + A_1' x + A_2' e^{2x} + A_3' x e^{2x} + F_1' + F_2' \tag{IV-5}$$

In the above equation, F_1' and F_2' are the particular solutions of F_I'. They are due to deflection w_I' and the temperature gradient T, respectively, i.e.,

$$\nabla^4 F_1' = \left(\frac{r_1}{h} \cot\beta\right) e^{-3x} (w_{,xx}' - w_{,x}') \tag{IV-6}$$

and

$$\nabla^4 F_2' = -\frac{\gamma_1^2}{h^2}\nabla^2(\alpha T) \tag{IV-7}$$

Substituting Eq. (IV-3) into Eq. (IV-6) and integrating it, we find

$$F_1' = \left(\frac{\gamma_1}{h}\cot\beta\right)e^{(1+\lambda)x}\sum_{i=1}^{M} a_i'\left(c_i' \sin\frac{i\pi x}{x_0} + d_i' \cos\frac{i\pi x}{x_0}\right) \tag{IV-8}$$

where c_i and d_i (i = 1, 3, 5) are found as functions of x_0 and are given explicitly in Appendix A.

The temperature gradient is chosen as an exponential function of x,

$$T_x = T_1 + \tau_1 e^{gx} \tag{IV-9}$$

where T_1, τ_1 and g are constants. If g vanishes, Eq. (IV-9) implies that the distribution of temperature is uniform over the surface of the conical shell. Integrating Eq. (IV-7), the function F_2' is obtained as:

$$F_2' = -\left[\left(\frac{\gamma_1^2}{h^2}\alpha\tau_1\right)\frac{1}{(2+g)^2}\right]e^{(2+g)x} \tag{IV-10}$$

Substituting Eqs. (IV-8), (IV-10) and (IV-5) into Eqs. (III-6), (III-7) and (IV-2), we obtain three algebraic equations. By solving these three equations, and the arbitrary constant A_0'

in Eq. (IV-5) is considered zero, the constants A_i' (i = 1, 2, 3) are found as:

$$A_i' = A_i'\left(x_0, \frac{r_i^2}{h^2}\alpha T_1, \frac{r_i^2}{h^2}\alpha \tau_1, a_1'\frac{r_1}{h}\cot\beta\right.$$

$$\left. a_3'\frac{r_1}{h}\cot\beta, a_5'\frac{r_1}{h}\cot\beta\right)$$

i = 1, 2, 3 (IV-11)

By substituting Eqs. (IV-3) and (IV-5) into Eq. (III-9), it is found that

$$D\nabla^4\left[\sum_{i=1}^{M} e^{\lambda x} a_i' \sin\frac{i\pi x}{x_0}\right] - \left(\frac{r_1}{h}\cot\beta\right)e^{-3x}$$

$$\times\left(F_{I,xx}' - F_{I,x}'\right) = \Delta Q_1$$

(IV-12)

Applying Galerkin's integration to Eq. (IV-12), and if only three terms of the prebuckling deflection are taken in Eq. (IV-3), the following three algebraic equations are obtained:

$$\int_0^{2\pi\sin\beta} \int_0^{x_0} \left[\Delta Q_1 e^{(2+\lambda)x} \sin\frac{i\pi x}{x_0}\right] dx\, d\theta = 0$$

i = 1, 3, 5 (IV-13)

Solving the above three equations simultaneously, the coefficients a_i' (i = 1, 3, 5) are obtained, which vary linearly with αT_1 and $\alpha\tau_1$. They can be expressed as:

$$a_i' = a_i'\left(x_o, \frac{r_1^2}{h^2}\alpha T_1, \frac{r_1^2}{h^2}\alpha \tau_1, A_1', A_2', A_3', \frac{r_1}{h}\cot\beta\right)$$

$$i = 1, 3, 5 \qquad \text{(IV-14)}$$

The six unknowns A_i' (i = 1, 2, 3) and a_i' (i = 1, 3, 5) can be determined by solving Eqs. (IV-11) and (IV-14). They are found to be functions of x_o, αT_1, $\alpha \tau_1$, $\frac{r_1}{h}$, and β. Therefore, F_I' is determined in accordance with the temperature distribution T. The expression A_i' and a_i' are given in Appendix A.

An additional deflection function during buckling which satisfies the simply supported boundary condition is assumed as:

$$w_I'' = e^{\lambda x}\left(a_1'' \sin\frac{\mu\pi x}{x_o} + a_3'' \sin\frac{3\mu\pi x}{x_o}\right)\cos\frac{\eta}{2\sin\beta}\theta \qquad \text{(IV-15)}$$

where μ and η are the numbers of half waves in meridional and circumferential directions, respectively. By substituting Eq. (IV-15) into Eq. (III-10) and letting

$$n = \frac{\eta}{2\sin\beta} \qquad \text{(IV-16)}$$

we obtain

$$\nabla^4 F_I'' = -\left(\frac{\gamma_1}{h}\cot\beta\right)e^{-3x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\left[e^{\lambda x}\right.$$

$$\left.\times\left(a_1'' \sin\frac{\mu\pi x}{x_o} + a_3'' \sin\frac{3\mu\pi x}{x_o}\right)\cos n\theta\right] \qquad \text{(IV-17)}$$

The additional stress function during buckling is the summation of two solutions: one is the homogeneous portion of Eq. (III-10), and the other is a particular solution which satisfies Eq. (IV-17). This stress function can be expressed as:

$$F_I'' = \left(A_1'' e^{-nx} + A_2'' e^{(2+n)x} + A_3'' e^{(2-n)x}\right)\cos n\theta$$

$$+\left(\frac{\gamma_1}{h}\cot\beta\right)e^{(1+\lambda)x}\sum_{i=1,3} a_i''\left(c_i'' \sin\frac{i\mu\pi x}{x_o} + \cos\frac{i\mu\pi x}{x_o}\right)\cos n\theta \qquad \text{(IV-18)}$$

where c_i'' and d_i'' (i = 1, 3) are found as functions of η, μ and x_o. The constants A_1'', A_2'' and A_3'' are determined in terms of a_i'' (i = 1, 3) from the following conditions of constraint::

$$F_{I,xx}'' - (1+\nu)F_{I,x}'' - \nu F_{I,\theta\theta}'' = 0$$

$$\text{at} \quad x = 0 \quad \text{and} \quad x = x_o \qquad \text{(IV-19)}$$

and

$$\int_0^{x_o}\left[(1+\nu)F_{I,x}'' - \nu F_{I,xx}'' + F_{I,\theta\theta}''\right]e^{-\alpha x}dx = 0 \qquad \text{(IV-20)}$$

Since it has been noted that the homogeneous solution of the stress function F has been already considered in the prebuckling state, we can say that the homogeneous solution of function F_2'' can be neglected without significant error [18].

When substituting Eqs. (IV-3), (IV-5), (IV-15) and (IV-8) into Eq. (III-11), it is found that

$$D\nabla^4 w_I'' = \left(\frac{\gamma_1}{h}\cot\beta\right)e^{-3x}(F_{I,xx}'' - F_{I,x}'') + e^{-4x}$$

$$\times (F_{I,xx}' - F_{I,x}')\left[(w_I' + w_I'')_{,x} + (w_I' + w_I'')_{,\theta\theta}\right]$$

$$-2e^{-4x}(F_{I,x\theta}' - F_{I,\theta}')\left[(w_I' + w_I'')_{,x\theta} - (w_I' + w_I'')_{,\theta}\right]$$

$$+e^{-4x}(F_{I,\theta\theta}' + F_{I,x}')\left[(w_I' + w_I'')_{,xx} - (w_I' + w_I'')_{,x}\right] = \Delta Q_2 \tag{IV-21}$$

By applying the Galerkin method to the above equation, a set of algebraic equations is then established:

$$\int_0^{x_0}\int_0^{2\pi\sin\beta} e^{(2+\lambda)x}\,\Delta Q_2 \sin\frac{i\mu x}{x_0}\cos n\theta\, dx\, d\theta = 0$$

$$i = 1, 3 \tag{IV-22}$$

After integration, Eq. (IV-22) can be briefly expressed in the following forms:

$$(c_{11} + d_{11}\alpha T_1 + f_{11}\alpha \tau_1)\, a_1'' + (c_{13} + d_{13}\alpha T_1 + f_{13}\alpha \tau_1)\, a_3'' = 0 \qquad \text{(IV-23)}$$

and

$$(c_{31} + d_{31}\alpha T_1 + f_{31}\alpha \tau_1)\, a_1'' + (c_{33} + d_{33}\alpha T_1 + f_{33}\alpha \tau_1)\, a_3'' = 0 \qquad \text{(IV-24)}$$

Eqs. (IV-23) and (IV-24) are two linear homogeneous, algebraic equations and have a nontrivial solution only if the determinant of the coefficients of a_1'' and a_3'' vanisnes. This requirement can be expressed as

$$[c_{11} + (d_{11} + f_{11}\gamma_1)\alpha T_1][c_{33} + (d_{33} + f_{33}\gamma_1)\alpha T_1] - [c_{13} + (d_{13} + f_{13}\gamma_1)\alpha T_1][c_{31} + (d_{31} + f_{31}\gamma_1)\alpha T_1] = 0 \qquad \text{(IV-25)}$$

where

$$\gamma_1 = \frac{\tau_1}{T_1} \qquad \text{(IV-26)}$$

Eq. (IV-25) is a second-order algebraic equation of T_1 and γ_1. For given values of γ_1 and g, two solutions of T_1 are

determined from Eq. (IV-25), and T_1 is found as a function of the number of half waves μ and η. After minimizing the solution of T_1, we find two values of the critical temperature. One of them is disregarded because it is physically impossible.

It has been observed that the buckling hoop stress is localized near the fixed edges; in other words, the hoop stress is high near the supports and it is low in the middle of the shell unless the shell is very short and thick [25]. This means that high average thermal hoop stress exists only when the shell is very stable, and the shell that buckles easily does not develop the hoop stress. For the buckling problem, it is found that the thermal hoop stress converges rapidly when a series in Eqs. (IV-3) and (IV-15) are used. It is also noted that the meridional compression has a much higher effect on the buckling temperature than the hoop stress does. Therefore, the value of the buckling temperature will not change significantly if more terms in Eqs. (IV-3) and (IV-15) are taken. Because the buckling is mainly caused by axial thermal compression, the conical shell is expected to buckle in multiple wave patterns with nearly the same wave length in both principal directions. The numerical calculations are thus made by taking $\mu/n = (H \tan \beta)/\pi R$ in Eq. (IV-25), where H and R represent the height and mean radius of the conical shell as shown in Fig. 2, page 25.

The results of the problem were obtained from the IBM 709 Computer, and are illustrated in Figs. 3 - 6. The relations presented in Figs. 3 - 5 show the effects of a conical shell's geometrical variables

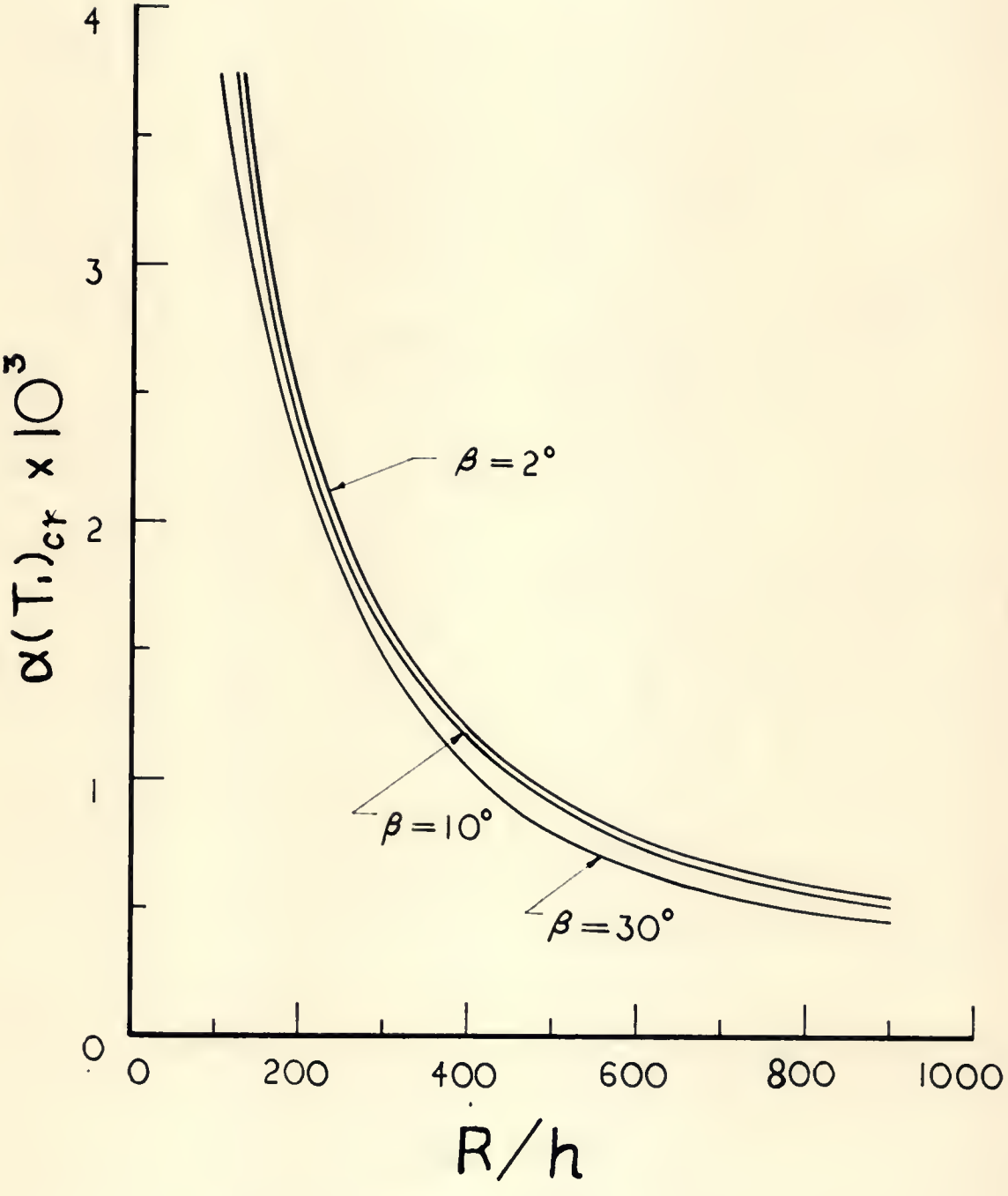


Fig. 3. Critical temperature versus radius-thickness ratio at H/R = 2.

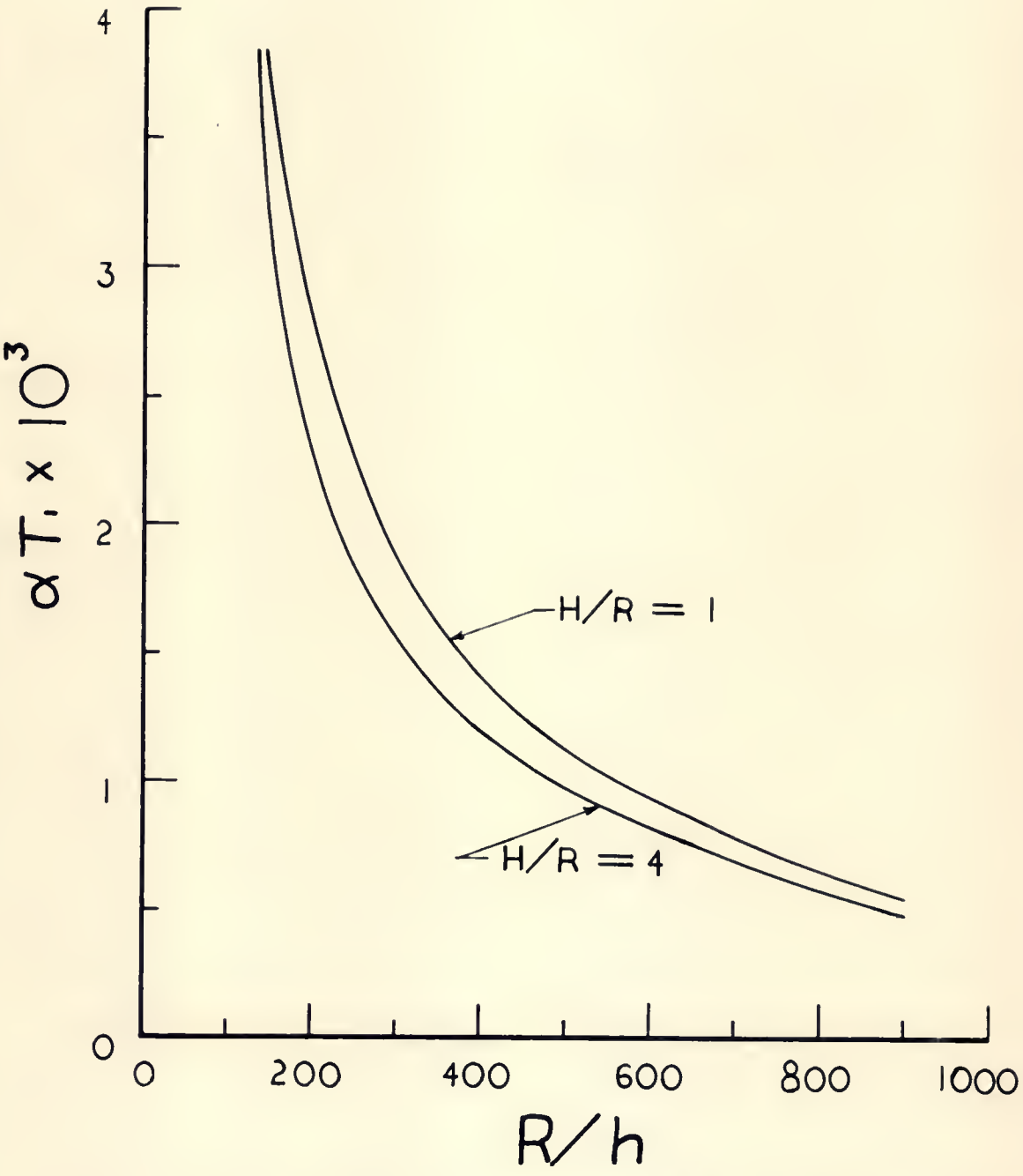


Fig. 4. Critical temperature versus radius-thickness ratio at $\beta = 10^{\circ}$.

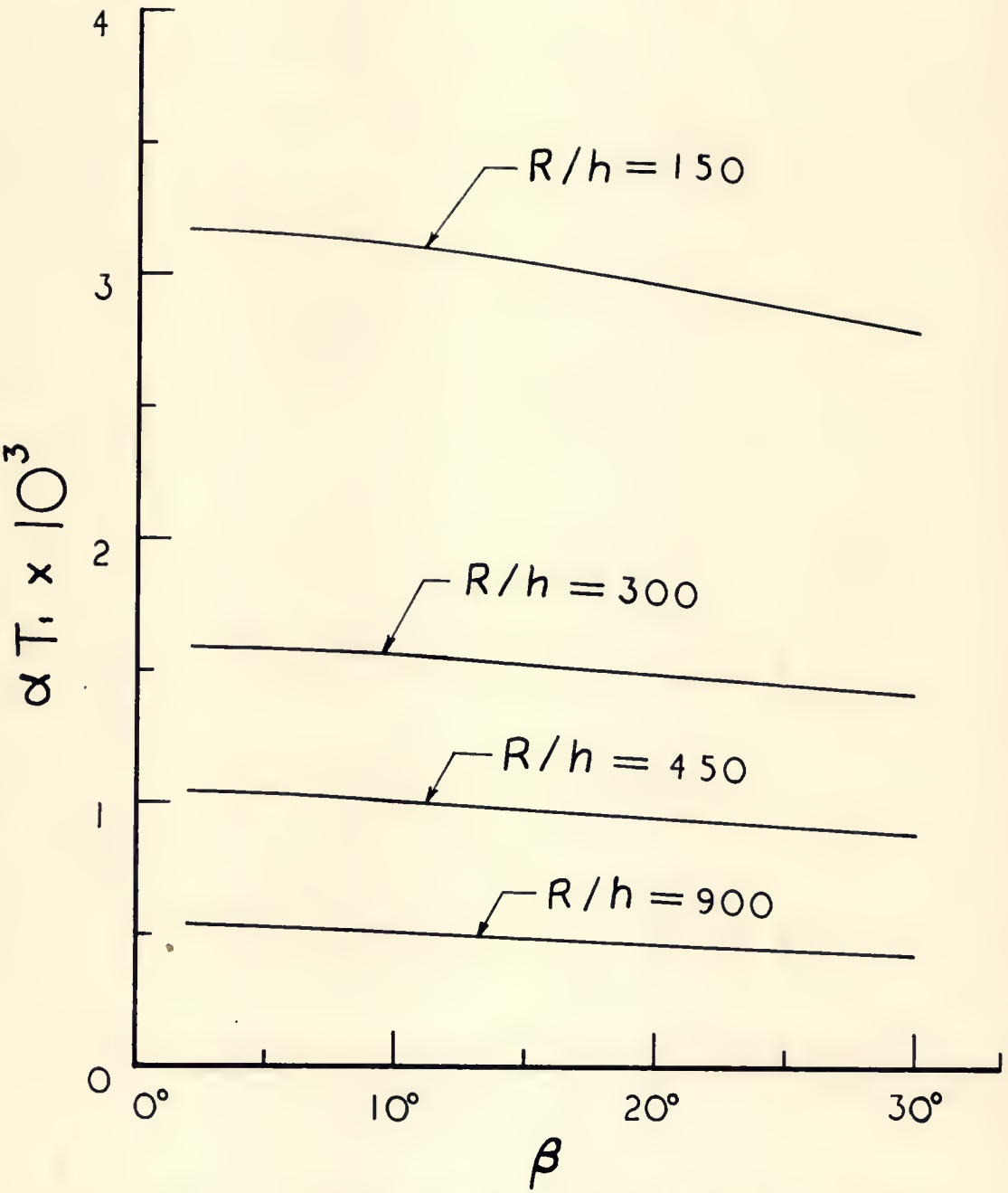


Fig. 5. Critical temperature versus semivertex angle at H/R = 2.

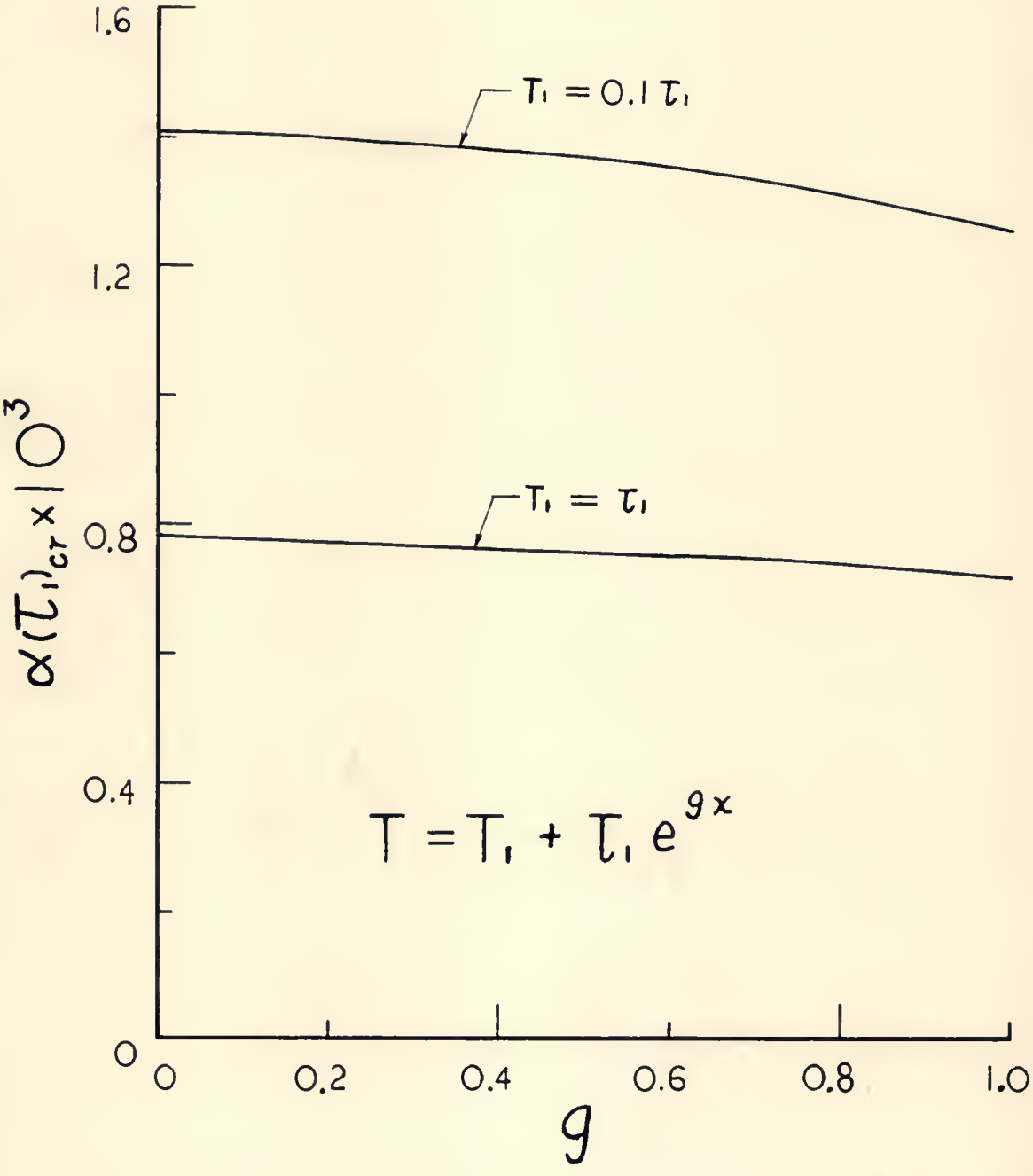


Fig. 6. Variation of critical temperature with meridional temperature index ($\beta = 10^{\circ}$, $R/n = 300$, $H/R = 2$).

on the critical value of uniform temperature rise (i.e., $\tau_1 = 0$). In Fig. 6, the variation of the critical temperature with the temperature index g of Eq. (IV-9) is depicted at a different τ_1/T_1 ratio.

Case II

In this case the conical shell is subjected to meridional and circumferential gradients and restricted circumferentially at both ends. Thus, in addition to the boundary condition expressed from Eqs. (III-4) to (III-7), other boundary conditions are

$$\int_0^{2\pi \sin\beta} \sigma_r \, d\theta = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = x_0 \qquad \text{(IV-27)}$$

or in another expression,

$$\int_0^{2\pi \sin\beta} (F_{,x} + F_{,\theta\theta}) \, d\theta = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = x_0 \qquad \text{(IV-28)}$$

The above conditions are applied to a shell which is unstrained in compression but is restrained in bending. If the temperature gradient has the form

$$T = \tau e^{-x} \sin^j\left(\frac{\theta}{2}\right)$$

(j = positive integer, $0 \le \theta \le 2\pi$)

the conical shell is hotter at one side than the other. As the index j increases, the heated portion becomes narrower and the buckling behavior is closer to the case under compression [35]. In the present study the temperature distribution is taken as:

$$T_{II} = T_2 + \tau_2 e^{-x} \sin k\theta \tag{IV-29}$$

with $k = 1/2 \sin \beta$.

The subscript II is used to indicate the functions associated with Case II. Both T_2 and τ_2 are taken as constants.

The normal deflection in the prebuckling state satisfies the simply supported boundary condition and is assumed in the form:

$$w_{II}' = \sum_{i=1}^{M} e^{\lambda x} (b_i' + b_{i0}' \cos n\theta) \sin \frac{i\pi x}{x_0} \tag{IV-30}$$

In this analysis, it is assumed that the prebuckling deflection w_{II}' is axisymmetric and with i = 1, 3, 5, takes the form,

$$w_{II}' = e^{\lambda x} \left(b_1' \sin \frac{\pi x}{x_0} + b_3' \sin \frac{3\pi x}{x_0} + b_5' \sin \frac{5\pi x}{x_0} \right) \tag{IV-31}$$

In a similar manner to the determination of stress function F_I' in Case I, the stress function F_{II}' prior to buckling can be found in the following form:

$$F_{II}' = F_3' + F_4' + F_5' \tag{IV-32}$$

where F_5' is the homogeneous solution of F_{II}' and is found as:

$$F_5' = B_1'(x^2 - \theta^2) + B_2' x + B_3' e^{2x} + B_4' x e^{2x} \tag{IV-33}$$

B_i' (i = 1, 2, 3, 4) can be determined by the conditions of constraints given in Eqs. (III-6), (III-7) and (IV-28). They are found to be functions of x_o, T_2, τ_2, $\frac{r_1}{h}$, β and b_i' with i = 1, 3, 5. The functions F_3' and F_4' in Eq. (IV-32) are the particular solutions of the stress function F_{II}', which correspond to the deflection function w_{II}' and the temperature gradient T, respectively. They can be expressed in the following forms:

$$F_3' = \left(\frac{r_1}{h}\cot\beta\right) e^{(1+\lambda)x} \sum_i b_i' \left(g_i' \sin\frac{i\pi x}{x_o} + h_i' \cos\frac{i\pi x}{x_o}\right) \qquad \text{(IV-34)}$$

and

$$F_4' = e^x (K-1)^{-1} \sin K\theta \left(\frac{r_1^2}{h^2}\alpha T_2\right) \qquad \text{(IV-35)}$$

The b_i''s (i = 1, 3, 5) are found by substituting Eqs. (IV-32), (IV-33), (IV-34) and (IV-35) into Eq. (III-9) and integrating it by the Galerkin method. The coefficients B_j' (j = 1, 2, 3, 4), g_i', h_i' and b_i' (i = 1, 3, 5) are functions of x_o, T_2, τ_2, $\frac{r_1}{h}$ and β, which are expressed explicitly in Appendix B.

It is noted in the above analysis, that the prebuckling hoop stress is independent of the temperature gradient provided the temperature function is chosen as in Eq. (IV-29), since

$$F_{4,xx}' - F_{4,x}' = 0 \qquad \text{(IV-36)}$$

During buckling, the additional deflection is assumed in the form:

$$w_{II}'' = e^{\lambda x}\left(b_1'' \sin\frac{\mu\pi x}{x_0} + b_3'' \sin\frac{3\mu\pi x}{x_0}\right)\cos n\theta \sin k\theta \tag{IV-37}$$

It has been mentioned in the previous case and reference [18] that the homogeneous solution of the stress function during buckling can be neglected without significant error. Therefore, only the particular solution of Eq. (III-10) is considered as the additional stress function during buckling.

By substituting Eq. (IV-37) into Eq. (III-10) and assuming that the stress function has the form

$$F_{II}'' = \left(\frac{\gamma_1}{h}\cot\beta\right) e^{(1+\lambda)x} \sum_{i=1,3} b_i''\left(g_i'' \sin\frac{i\mu\pi x}{x_0} + h_i'' \cos\frac{i\mu\pi x}{x_0}\right)\cos n\theta \sin k\theta \tag{IV-38}$$

and g_i'' and h_i'' are found by the use of the Galerkin method.

Substituting Eq. (IV-38) into Eq. (III-10), we have

$$\nabla^4\left[\left(\frac{\gamma_1}{h}\cot\beta\right) e^{(1+\lambda)x} \sum_{i=1,3} b_i''\left(g_i'' \sin\frac{i\mu\pi x}{x_0} + h_i'' \cos\frac{i\mu\pi x}{x_0}\right) \times \cos n\theta \sin k\theta\right] + \left(\frac{\gamma_1}{h}\cot\beta\right) e^{-3x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\left[e^{\lambda x} \sum_{i=1,3} b_i'' \times \sin\frac{i\mu\pi x}{x_0} \cos n\theta \sin k\theta\right] = 0 \tag{IV-39}$$

In order to determine the coefficients g_i'' and h_i'' (i = 1, 3), the following approach is employed; Eq. (IV-39) is integrated by the use of the Galerkin method in the circumferential direction, which provides the following relations:

$$\int_0^{2\pi \sin\beta} \Delta Q_3 \sin k\theta \cos n\theta \, d\theta = 0 \qquad \text{(IV-40)}$$

After integration, the coordinate parameter θ vanishes in the above equation, and it becomes a homogeneous equation in terms of x only. The coefficients g_i'' and h_i'' (i = 1, 3) are obtained from Eq. (IV-40) by comparing the corresponding terms of x. They are found to be functions of μ, η, x_o, k, $\frac{r_1}{h}\cot\beta$, and are given in Appendix B.

Substituting Eqs. (IV-37) and (IV-38) into Eq. (III-11), two algebraic equations are obtained by applying the Galerkin method to Eq. (III-11). For this case, $w'' = w_{II}''$, $F' = F_{II}'$, $F'' = F_{II}''$ and $w = w_I' + w_{II}''$ in Eq. (III-11). If the following notation is introduced

$$\gamma_2 = T_2/\tau \qquad \text{(IV-41)}$$

the two algebraic equations can be briefly expressed as:

$$(i_{11} + k_{11}\alpha\tau_2)b_1'' + (i_{13} + k_{13}\alpha\tau_2)b_3'' = 0 \qquad \text{(IV-42)}$$

$$(i_{31} + k_{31}\alpha\tau_2)b_1'' + (i_{33} + k_{33}\alpha\tau_2)b_3'' = 0 \qquad \text{(IV-43)}$$

For a nontrivial solution, the determinant of Eqs. (IV-42) and (IV-43) is zero. After solving the determinant, the term τ_2 of the temperature gradient can be written as:

$$\alpha\tau_2 = \alpha\tau_2\left(\kappa, \mu, \eta, \frac{r}{h}, \beta, \gamma_2\right) \qquad \text{(IV-44)}$$

In a similar manner to the determination of critical temperature in Case I, the critical temperature in this case is found by minimizing the wave numbers μ and η of Eq. (IV-44). The critical temperature at different magnitudes of γ_2 is shown in Fig. 7. The details of expressions used in this case are given in Appendix B.

2. Minimum Temperature

After buckling the normal deflection becomes finite; thus the second-degree terms of the derivatives of normal deflection should be included in the geometric (strain-displacement) relations. These relationshhips are expressed in Eqs. (II-18), and the nonlinear equations (II-37) and (II-39) will be used for the solution. Since the thermal stresses in the shell depend on the boundary restraint, unlike the

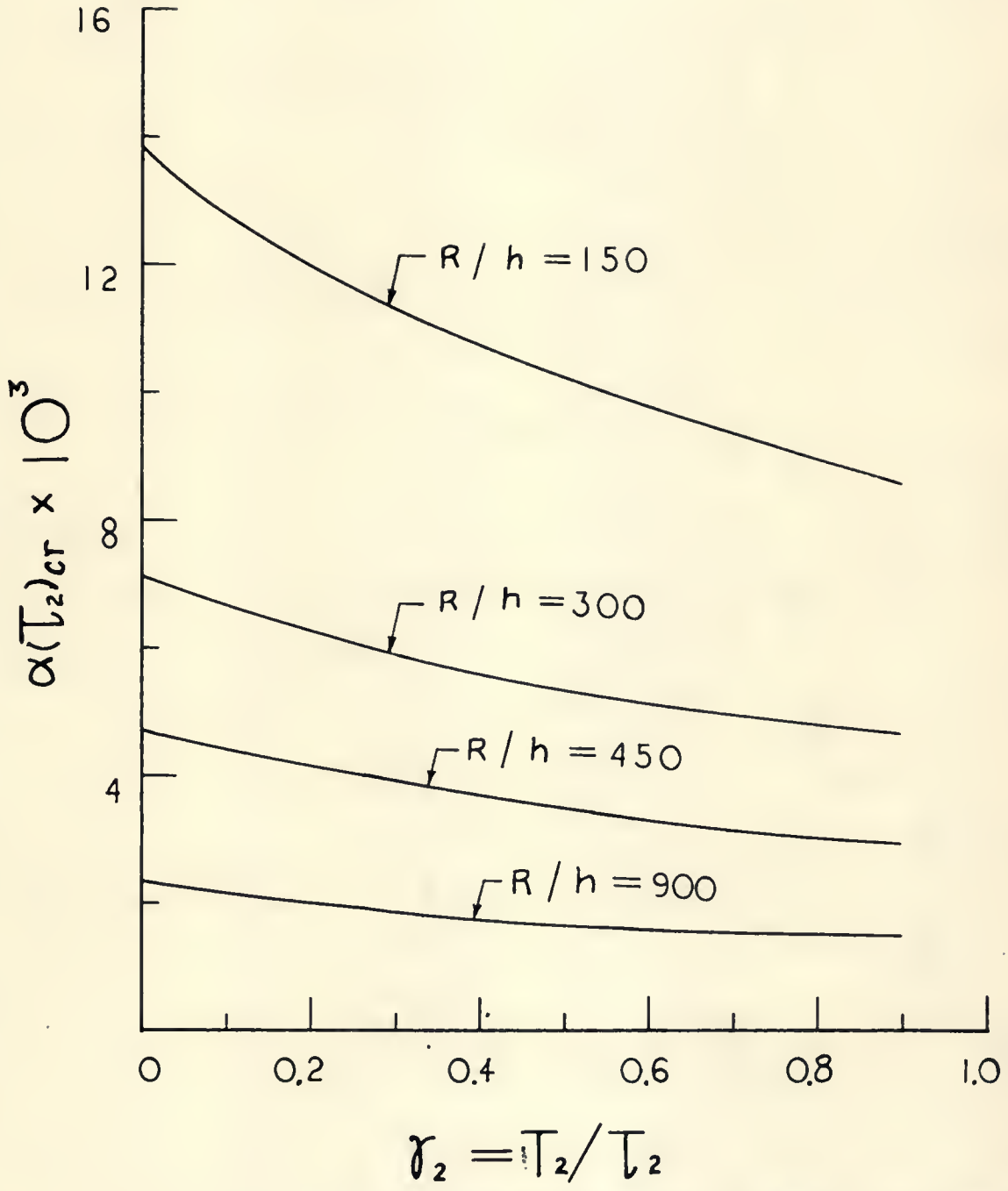


Fig. 7. Critical temperature due to circumferentially non-uniform heating ($\beta = 10^{\circ}$, H/R = 2).

case under external loading, the average membrane stresses will be different before and after deformation. The nonlinear effect on the value of the temperature gradient to maintain equilibrium after buckling, is here investigated. In the present nonlinear analysis, the temperature change and boundary conditions are the same as considered in Case I in the linear analysis of the buckling problem, in which the conical shell is subjected to meridional temperature gradient and restricted completely at both ends. In order to compare the value of buckling temperature with the minimum temperature in equilibrium state after buckling, the deflection functions used in the nonlinear analysis are basically tne same as those assumed in the linear case, but only two terms are used in the prebuckling state as an approximation.

Another boundary condition in addition to Eqs. (III-6) and (III-7) is,

$$\int_0^{2\pi \sin\beta} \int_0^{x_0} u_{,x}\, d\theta\, dx = 0 \qquad \text{(IV-45)}$$

The above condition implies that the average length of the cone is unchanged during heating. Eq. (IV-45) can be expressed in terms of the stress function F and the normal deflection w,

$$\int_0^{2\pi \sin\beta} \int_0^{x_0} \Big\{ e^{-x}\left[(1+\nu)F_{,x} + F_{,\theta\theta} - \nu F_{,xx}\right] + \frac{\gamma^2}{h^2}\alpha T e^{x} - \frac{1}{2} e^{-x}(w_{,x})^2 \Big\} d\theta\, dx = 0 \qquad \text{(IV-46)}$$

It can be observed from the above condition that for the same temperature rise, the average stress is less than that considered in the linear case, because the additional nonlinear term in Eq. (IV-47) is always negative.

The normal deformation prior to buckling is chosen as:

$$w' = e^{\lambda x}\left(\hat{a}_1' \sin\frac{\pi x}{x_0} + \hat{a}_3' \sin\frac{3\pi x}{x_0}\right) \qquad \text{(IV-47)}$$

In a similar manner to the determination of the prebuckling stress function in the linear analysis, the stress function prior to buckling is

$$F' = \hat{A}_1' x + \hat{A}_2' e^{2x} + \hat{A}_3' x e^{2x} + F_5' + F_6' \qquad \text{(IV-48)}$$

The boundary condition and the temperature distribution in this nonlinear analysis are the same as in Case I; therefore, by setting the terms to correspond to the third term of the deflection function, w_I' equals zero, the following relations are obtained:

$$F_5' = F_1', \quad F_6' = F_2', \quad \hat{A}_1' = A_1'$$
$$\hat{A}_2' = A_2', \quad \hat{A}_3' = A_3' \qquad \text{(IV-49)}$$

The additional deformation during buckling is assumed the same as in Eq. (15), i.e.,

$$w'' = e^{\lambda x}\left(\hat{a}_1'' \sin\frac{\mu\pi x}{x_0} + \hat{a}_3'' \sin\frac{3\mu\pi x}{x_0}\right)\cos n\theta \qquad \text{(IV-50)}$$

The additional stress function during buckling has the form:

$$F'' = \hat{A}_1'' x + \hat{A}_2'' e^{2x} + \hat{A}_3'' x e^{2x} + F_5'' \qquad \text{(IV-51)}$$

The particular solution of F'' has been found as

$$F_5'' = e^{(1+\lambda)x}\left(\hat{a}_1'' K_1 \cos m_1'' x + \hat{a}_1'' K_2 \sin m_1'' x + \hat{a}_3'' K_3 \cos m_3'' x\right.$$

$$\left. + \hat{a}_3'' K_4 \sin m_3'' x\right)\cos n\theta + \hat{a}_1''\left(\frac{r_1^2}{h^2}\alpha T_1\right) e^{2\lambda x}\left[K_5 \cos(m_1'' - m_1')x\right.$$

$$+ K_6 \sin(m_1'' - m_1')x + K_7 \cos(m_1'' + m_1')x + K_8 \sin(m_1'' + m_1')x$$

$$+ K_9 \cos(m_1'' - m_3')x + K_{10} \sin(m_1'' - m_3')x + K_{11} \cos(m_1'' + m_3')x$$

$$\left. + K_{12} \sin(m_1'' + m_3')x\right]\cos n\theta + \hat{a}_3''\left(\frac{r_1^2}{h^2}\alpha T_1\right) e^{2\lambda x}$$

$$\times\left[K_{13} \cos(m_3'' - m_1')x + K_{14} \sin(m_3'' - m_1')x + K_{15} \cos(m_3'' + m_1')x\right.$$

$$+ K_{16} \sin(m_3'' + m_1')x + K_{17} \cos(m_3'' - m_3')x + K_{18} \sin(m_3'' - m_3')x$$

$$\left. + K_{19} \cos(m_3'' + m_3')x + K_{20} \sin(m_3'' + m_1')x\right]\cos n\theta + \left(\frac{r_1^2}{h^2}\alpha T_1\right)^2$$

$$\times e^{2\lambda x}\left[K_{21} + K_{22} \cos 2m_1' x + K_{23} \sin 2m_1' x + K_{24} \cos(m_1' + m_3')x\right.$$

$$\left. + K_{25} \sin(m_3' - m_1')x + K_{26} \cos(m_3' + m_1')x + K_{27} \sin(m_3' + m_1')x\right.$$

$$+ K_{28}\cos 2m_3'x + K_{29}\sin 2m_3'x\,] + \hat{a}_1''^2 e^{2\lambda x}[K_{30}$$

$$+ K_{31}\cos 2m_1''x + K_{32}\sin 2m_1''x + K_{33}\cos 2n\theta + K_{34}\cos 2n\theta$$

$$\times \cos 2m_1''x + K_{35}\cos 2n\theta \sin 2m_1''x\,) + \hat{a}_1''\,\hat{a}_3''\, e^{2\lambda x}$$

$$\times [K_{36}\cos(m_3''-m_1'')x + K_{37}\sin(m_3''-m_1'')x + K_{38}\cos(m_1''+m_3'')x$$

$$+ K_{39}\sin(m_1''+m_3'')x + K_{40}\cos(m_3''-m_1'')x\cos 2n\theta + K_{41}\sin(m_3'-m_1')x$$

$$\times \cos 2n\theta + K_{42}\cos(m_3''+m_1'')x\cos 2n\theta + K_{43}\sin(m_3''+m_1'')x]$$

$$+ \hat{a}''^2 e^{2\lambda x}(K_{44} + K_{45}\cos 2m_3''x + K_{46}\sin 2m_3''x + K_{47}\cos 2n\theta$$

$$+ K_{48}\cos 2m_3''\cos 2n\theta + K_{49}\sin 2m_3''x\cos 2n\theta\,) \qquad \text{(IV-52)}$$

in which

$$m_i' = \frac{i\pi}{x_0} \quad , \quad m_i'' = \frac{i\mu\pi}{x_0} \qquad \text{(IV-53)}$$

i is an odd integer

By substituting Eqs. (IV-51) and (IV-52) into Eq. (III-12) and comparing the corresponding coefficients, K_j (j = 1, 2, ... 49) are

obtained as functions of x_o, Υ, μ, $\frac{r_1}{h}$, β and τ_1. They are given in Fortran language in Appendix C.

By applying the relation (IV-14) to Eqs. (III-6), (III-7) and (IV-46), the boundary conditions during buckling are

$$(F''_{,xx} - F''_{,x}) - \nu (F''_{,\theta\theta} + F''_{,x}) = 0$$

at $x = 0$ and $x = x_o$ (IV-54)

and

$$\int_0^{x_o}\int_0^{2\pi\sin\beta} e^{-x}\left\{[(1+\nu)F''_{,x} + F''_{,\theta\theta} - \nu F''_{,xx}] - \frac{1}{2}(w_{,x})^2\right\} dx\,d\theta = 0 \qquad \text{(IV-55)}$$

By substituting Eqs. (IV-50), (IV-51) and (IV-52) into Eqs. (IV-54) and (IV-55), we obtained three algebraic equations: $\hat{A}''_1$, $\hat{A}''_2$ and $\hat{A}''_3$ which are then obtained by solving these three equations in the following forms:

$$\hat{A}''_1 = g_1 \hat{a}''^2_1 + g_2 \hat{a}''_1 \hat{a}''_3 + g_3 \hat{a}''^2_3 + g_4\left(\frac{r_1^2}{h^2}\alpha T_1\right)$$

$$\hat{A}''_2 = g_5 \hat{a}''^2_1 + g_6 \hat{a}''_1 \hat{a}''_3 + g_7 \hat{a}''^2_3 + g_8\left(\frac{r_1^2}{h^2}\alpha T_1\right)$$

$$\hat{A}''_3 = g_9 \hat{a}''^2_1 + g_{10} \hat{a}''_1 \hat{a}''_3 + g_{11} \hat{a}''^2_3 + g_{12}\left(\frac{r_1^2}{h^2}\alpha T_1\right) \qquad \text{(IV-56)}$$

Substituting the expressions in Eq. (IV-47) - (IV-52) into the equation of equilibrium (III-13) and assuming the error is ΔQ_4,

two equations are obtained by applying the Galerkin method:

$$\int_0^{x_0}\int_0^{2\pi \sin\beta} \Delta Q_4 e^{(2+\lambda)x} \sin m_1'' x \cos n\theta \, dx \, d\theta = 0 \qquad \text{(IV-57)}$$

$$\int_0^{x_0}\int_0^{2\pi \sin\beta} \Delta Q_4 e^{(2+\lambda)x} \sin m_3'' x \cos n\theta \, dx \, d\theta = 0 \qquad \text{(IV-58)}$$

The above two equations can be expressed in the following forms after integration:

$$\begin{aligned} &\hat{a}_1'' R_1 + \hat{a}_3'' R_2 + \hat{a}_1''\left(\frac{r_i^2}{h^2}\alpha T_1\right) R_3 + \hat{a}_3''\left(\frac{r_i^2}{h^2}\alpha T_1\right) R_4 \\ &+ \hat{a}_1''\left(\frac{r_i^2}{h^2}\alpha T_1\right)^2 R_5 + \hat{a}_3''\left(\frac{r_i^2}{h^2}\alpha T_2\right) R_6 + \hat{a}_1''^3 R_7 \\ &+ \hat{a}_1''^2 \hat{a}_3'' R_8 + \hat{a}_1'' \hat{a}_3''^2 R_9 + \hat{a}_3''^3 R_{10} = 0 \end{aligned} \qquad \text{(IV-59)}$$

$$\begin{aligned} &\hat{a}_1'' S_1 + \hat{a}_3'' S_2 + \hat{a}_1''\left(\frac{r_i^2}{h^2}\alpha T_1\right) S_3 + \hat{a}_3''\left(\frac{r_i^2}{h^2}\alpha T_1\right) S_4 \\ &+ \hat{a}_1''\left(\frac{r_i^2}{h^2}\alpha T_1\right)^2 S_5 + \hat{a}_3''\left(\frac{r_i^2}{h^2}\alpha T_1\right) S_6 + \hat{a}_1''^3 S_7 \\ &+ \hat{a}_1''^2 \hat{a}_3'' S_8 + \hat{a}_1'' \hat{a}_3''^2 S_9 + \hat{a}_3''^3 S_{10} = 0 \end{aligned} \qquad \text{(IV-60)}$$

In an attempt to solve Eqs. (IV-59) and (IV-60) simultaneously, it is necessary to define

$$\delta = \frac{\hat{a}_1''}{\hat{a}_3''} \qquad \text{(IV-61)}$$

By substituting Eq. (IV-61) into Eqs. (IV-59) and (IV-60), and solving for the constant T_1, we find

$$\alpha T_1 = \frac{1}{2}\left[-V_2 \pm (V_2^2 - 4V_1V_3)^{0.5}\right] V_1^{-1} \frac{h^2}{r_1^2} \qquad \text{(IV-62)}$$

where v_1, v_2 and v_3 are functions of μ, η, $\frac{r_1}{h}$, β, ν_1 and ζ. If the geometry of the conical shell $\frac{R}{h}$, β and the temperature ratio ν_1 is given, the value of αT_1 can be plotted versus ζ for fixed values of μ and η. Examples are given in Figs. 8 and 9. The critical temperatures are then found from the minimum value of these curves. The temperatures should be equivalent to the values found from the relation $\partial T_1/\partial \zeta = 0$. The numerical results for this case are illustrated in Figs. 10 and 11.

3. Deformation

The relations between the temperature rise and the deformation after buckling can be obtained by solving Eqs. (IV-59) and (IV-60) simultaneously, in which $\hat{a}_1''$ and $\hat{a}_3''$ are expressed in terms of T_1 and ν_1.

Numerical examples are given for symmetrical (n = 0) as well as unsymmetrical cases. If only the first term of the deflection function w'' is taken, i.e., $\hat{a}_3''$ is equal to zero in Eq. (IV-50), then Eq. (IV-59) can be written as:

$$R_7 \hat{a}_1''^2 + \frac{r_1^2}{h^2}\alpha T_1\left[\left(\frac{r_1^2}{h^2}\alpha T_1\right) R_5 + R_3\right] = 0 \qquad \text{(IV-63)}$$

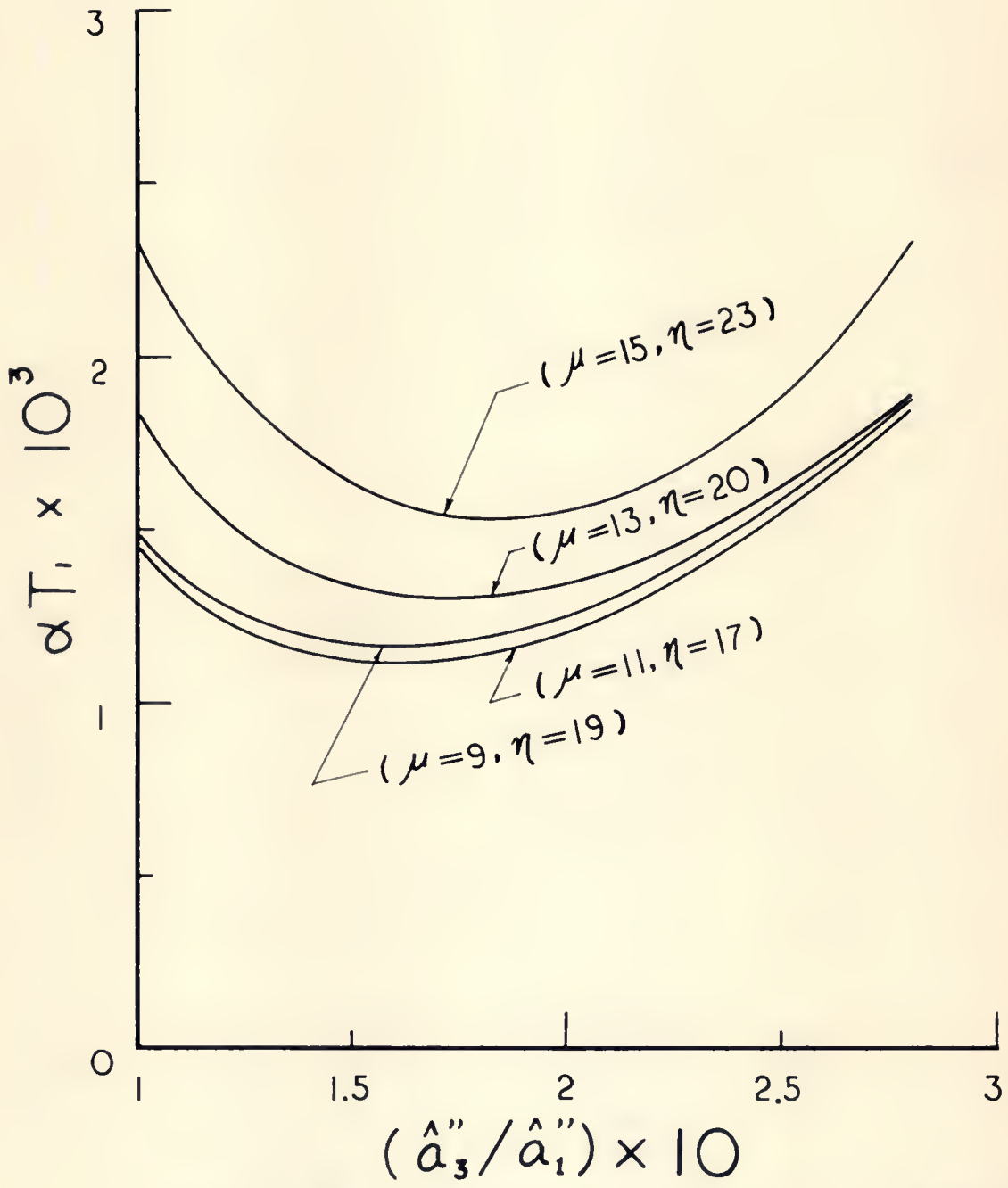


Fig. 8. Temperature variation T_1 as a function of deflection coefficient ratio $(\hat{a}_3''/\hat{a}_1'')$ at $\beta = 10^o$, R/h = 450 and H/R = 2.

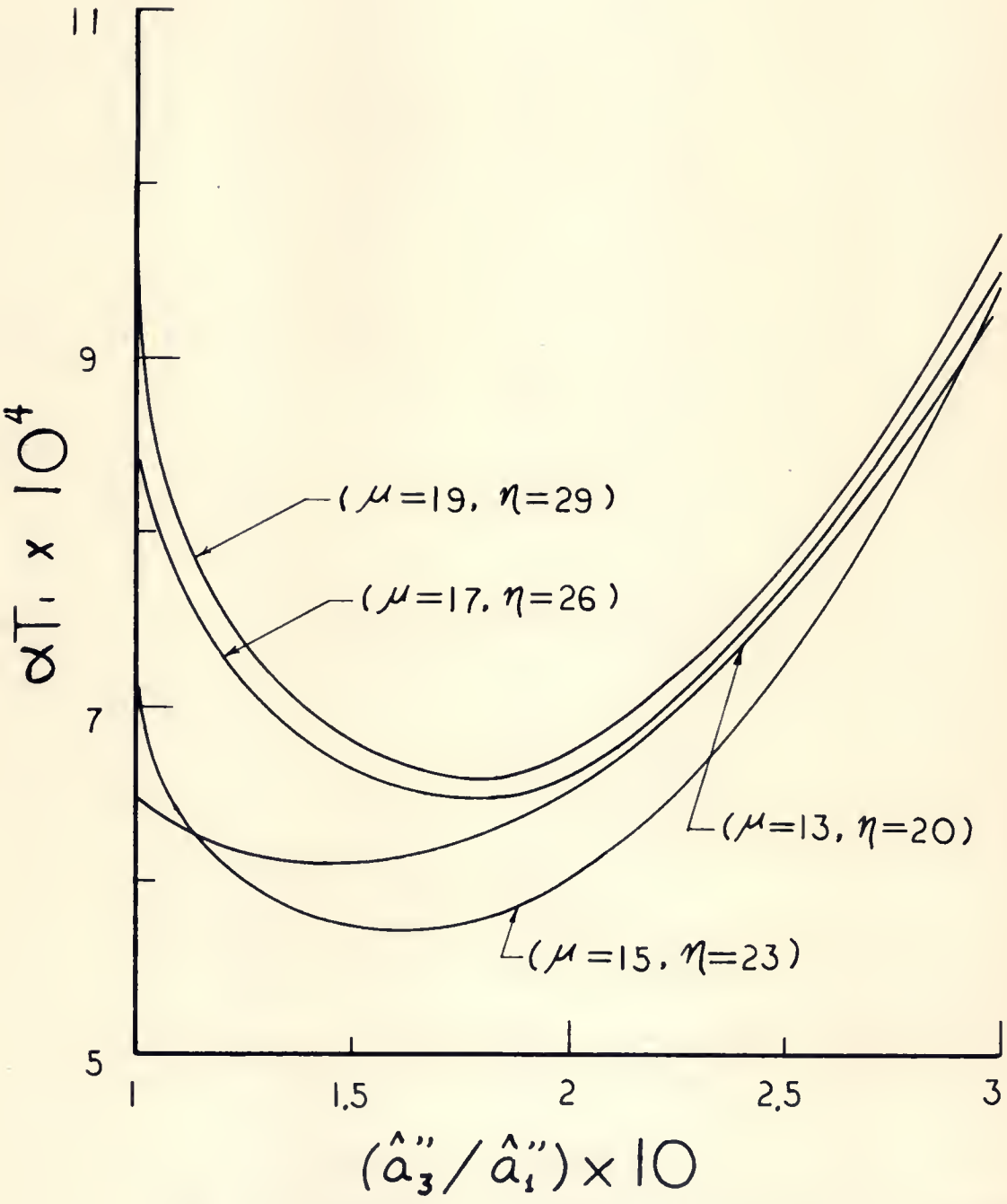


Fig. 9. Temperature variation T_1 as a function of deflection coefficient ratio $(\hat{a}_3''/\hat{a}_1'')$ at $\beta = 10^\circ$, $R/h = 900$ and $H/R = 2$.

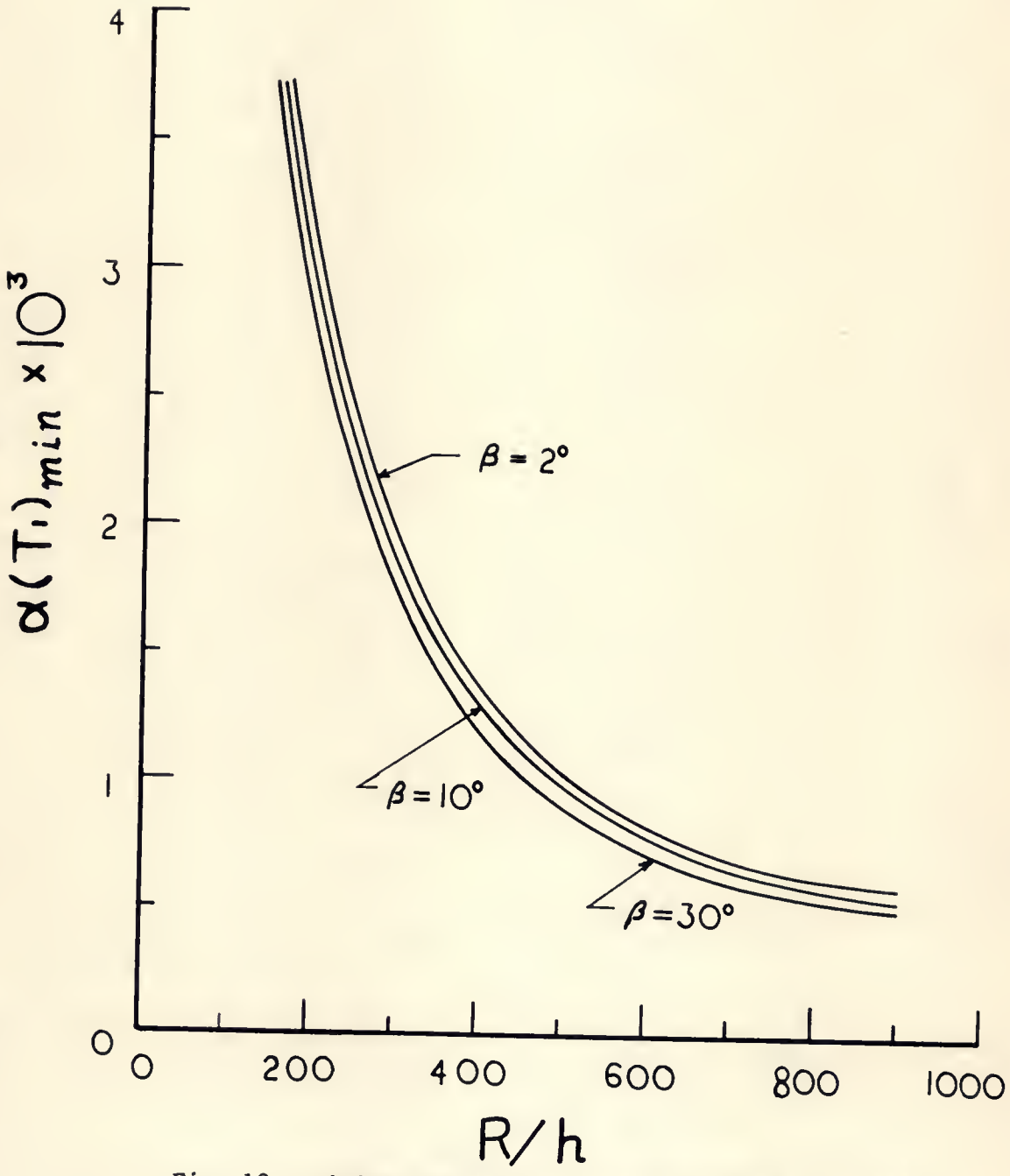


Fig. 10. Minimum temperature versus radius-thickness ratio at H/R = 2.

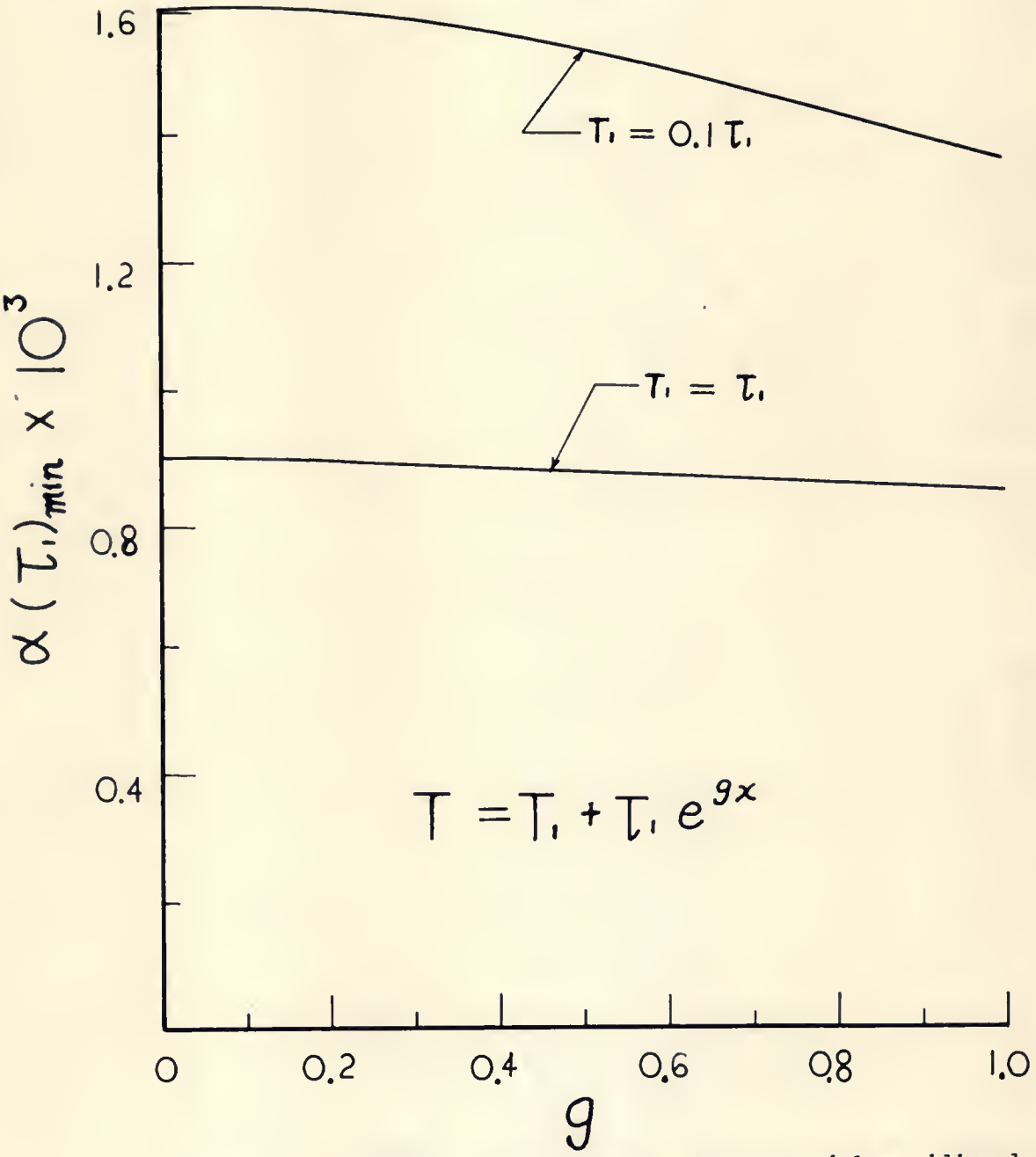


Fig. 11. Variation of minimum temperature with meridional temperature index ($\beta = 10^o$, R/h = 300, H/R = 2).

The above equation can be solved for the magnitude of the deformation $\hat{a}_1''$ in terms of T_1 and ν_1. From uniform temperature distribution, ν_1 is equal to zero. Figs. 12 and 13 give the value of T_i versus $(w'')_{max}$ for different combinations of wave numbers, while $\Delta T = T_1 - (T_1)_{cr}$ versus $(w'')_{max}$ curves are plotted in Fig. 14.

Since the numerical calculation is very cumbersome, the high-speed electronic computer is therefore employed. The solution programs were written in Fortran IV language, and the numerical work was carried out on the IBM 709 Computer at the University of Florida Computing Center.

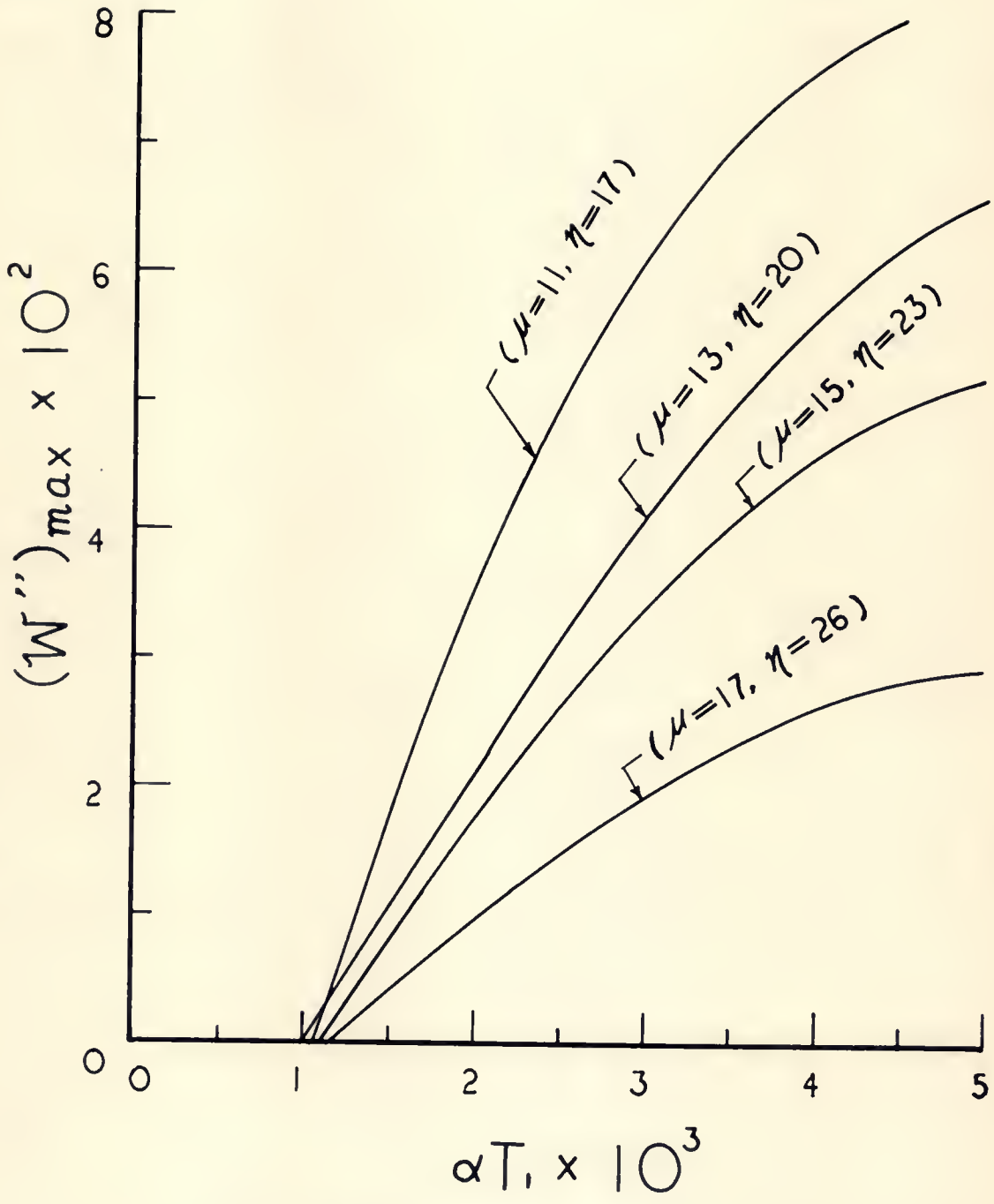


Fig. 12. Deflection versus temperature ($\beta = 10^{\circ}$, $R/h = 450$, $H/R = 2$).

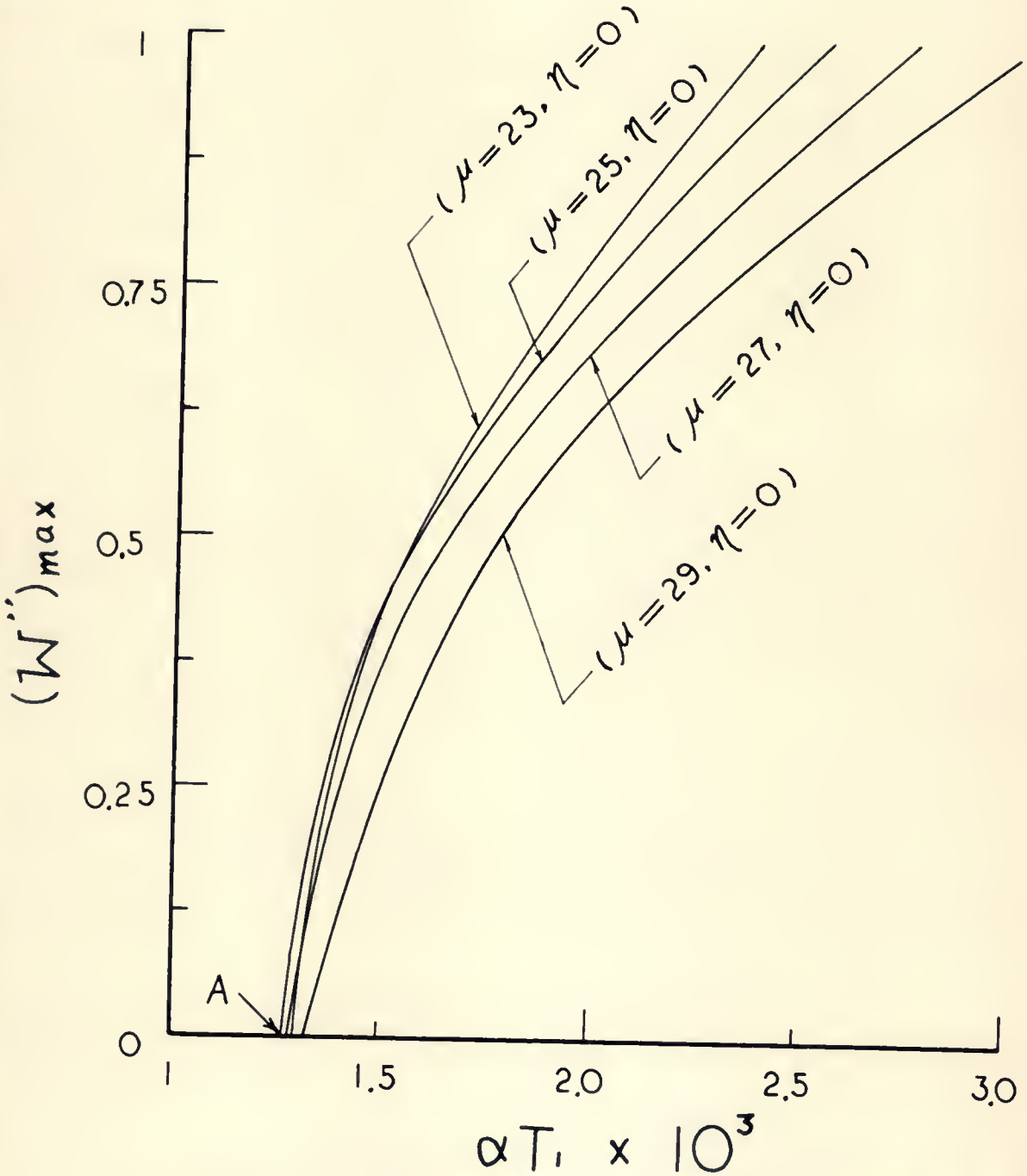


Fig. 13. Deflection versus temperature for axisymmetric case ($\beta = 10^{\circ}$, R/h = 450, H/R = 2).

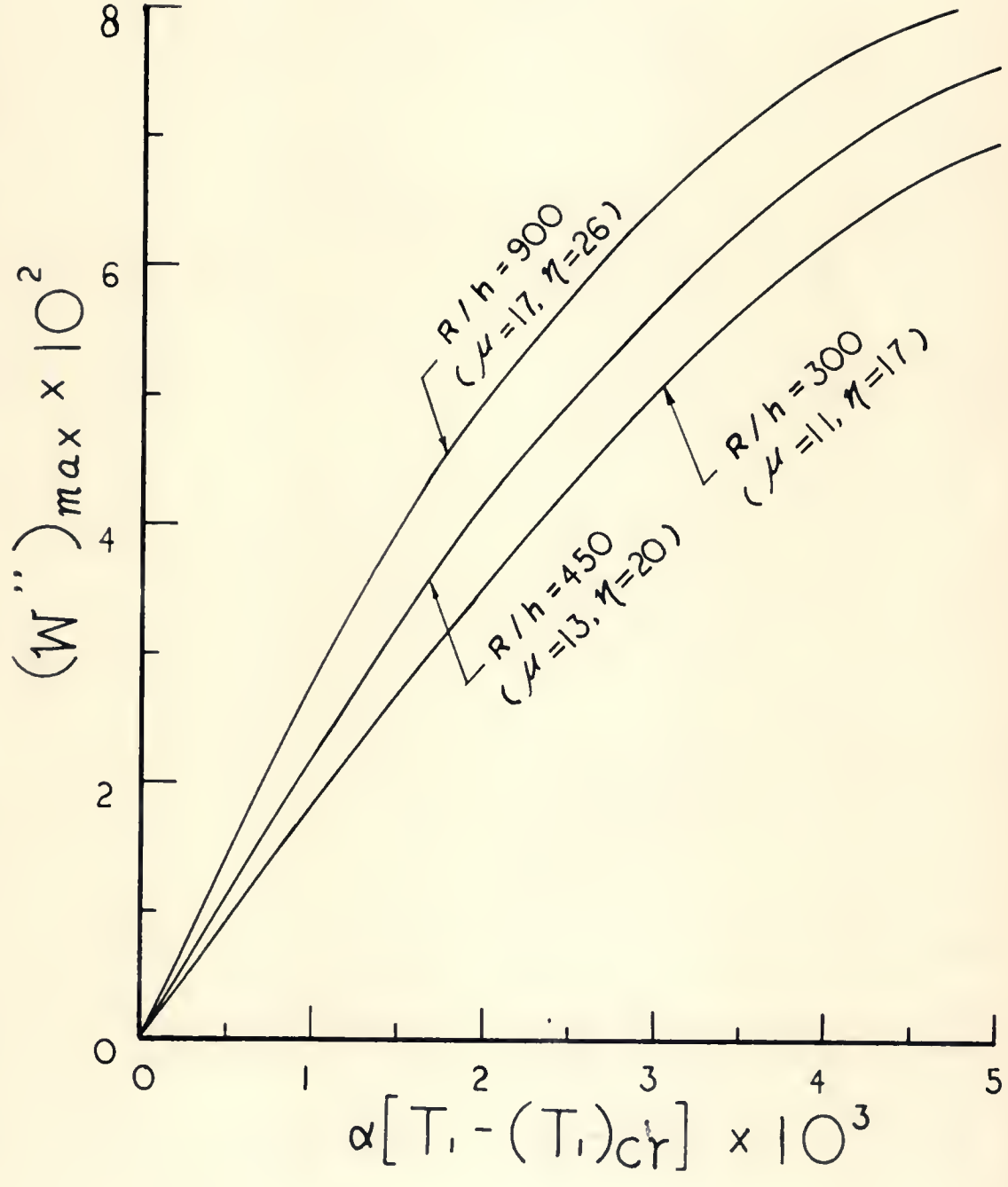


Fig. 14. Deflection versus temperature ($\beta = 10^{\circ}$, H/R = 2).

CHAPTER V

CONCLUSIONS

Numerical examples have been given in the previous chapter for different cases. It has been found in Figs. 3, 4, 5 and 10 that the radius-thickness ratio has the most effect on critical temperature, while the change of ratio H/R and the semivertex angle of the cone β vary the temperature only slightly. It can be observed from Figs. 3 and 7 that for the same geometrical parameters, the critical temperature in Case II is almost five times higher than in Case I. This implies that when the shell is longitudinally restrained, the axial stress plays a more important role than the hoop stress during buckling. Since the thermal stresses depend on the condition of constraint and temperature distribution, the thermal buckling problems have to be treated individually for each case. The two cases (Cases I and II) considered in this study are those under which the shell is the most likely to buckle.

Figs. 12 to 14 give the deformation-temperature relations. These figures indicate that the deformation is proportional to the temperature, the ratio R/h, and is inversely proportional to the numbers of waves in two principal directions. It can be seen from the condition of constraint Eq. (IV-44), that for the same temperature, the thermal stresses decrease with the increase of normal deformation.

However, in the axisymmetric case, the deformation after buckling is very sensitive to the increase of temperature, and it is also noted in Fig. 13 that point A should be the critical temperature for this case.

The buckling of shells subjected to external loadings has been studied by many authors in the past. They found that, in general, the buckling loads obtained by the use of the classical theory were larger than those found by means of the finite-deflection theory. It is interesting to note that in the present investigation a higher buckling temperature was found by nonlinear analysis than that obtained by linear analysis. This phenomenon can be explained by the condition of constraint Eq. (IV-1). When the large deflection theory is considered, the nonlinear term $(w_{,x})^2$ appears in the condition of constraint as expressed in Eq. (IV-44), since $(w_{,x})^2$ is always positive; therefore, it can be seen from Eq. (IV-44) that during buckling a certain amount of thermal stress is released by the consideration of this term. Consequently, in the use of the large deflection theory, a higher critical temperature is obtained in the buckling of shells. Thus, for the post-buckling case, the shell will remain stable at critical temperature and becomes unstable when the temperature reaches minimum temperature. A comparison of critical temperature with minimum temperature at $\beta = 10$ and $H/R = 2$ is given in Fig. 15.

In order to compare the experimental with the theoretical results, a test is performed in this study. The truncated cones under the test were fabricated from a flat brass sheet with a thickness of 0.005 inch, and the coefficient of thermal expansion is $10.4 \times 10^{-5}/^{o}F$.

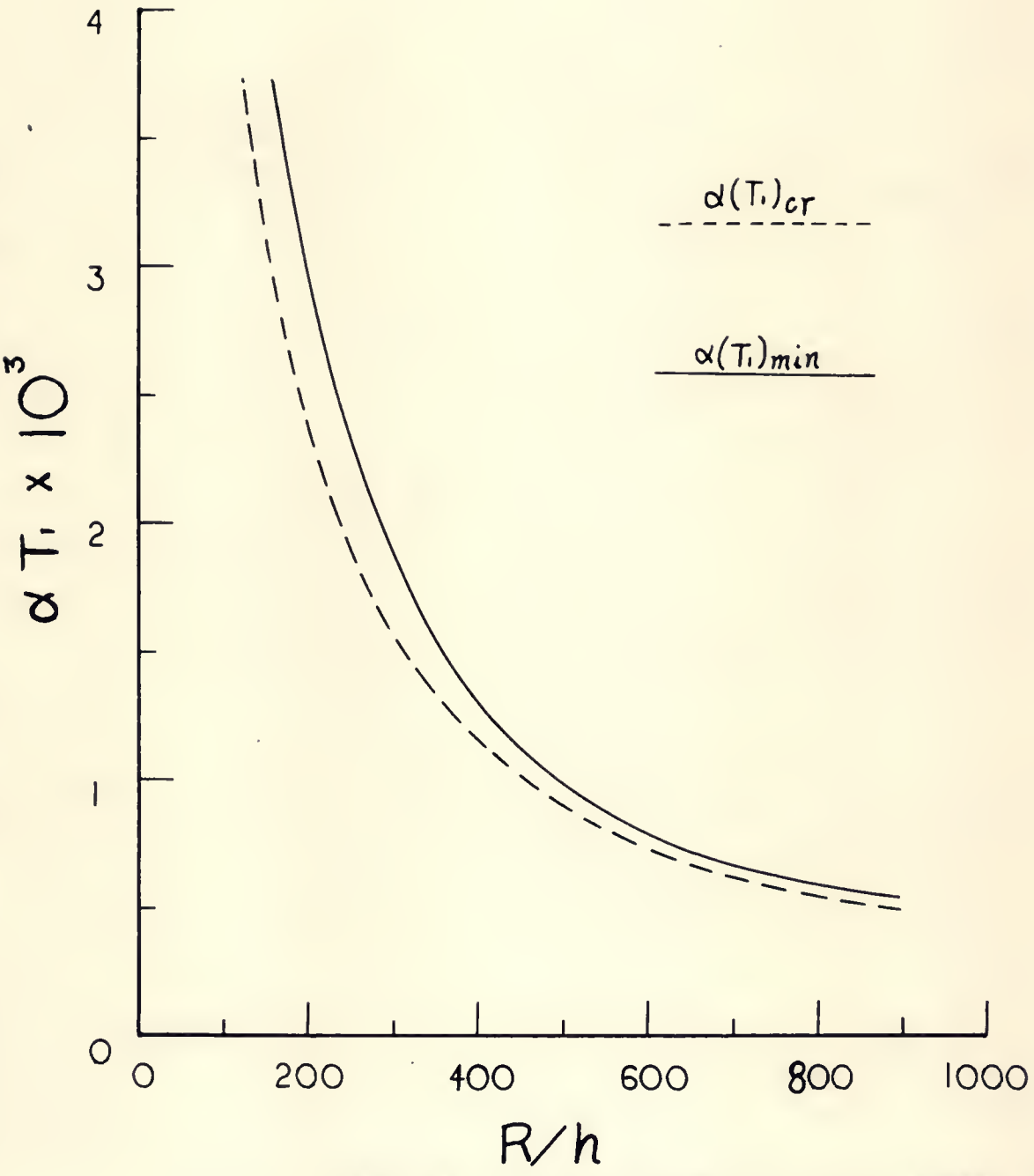
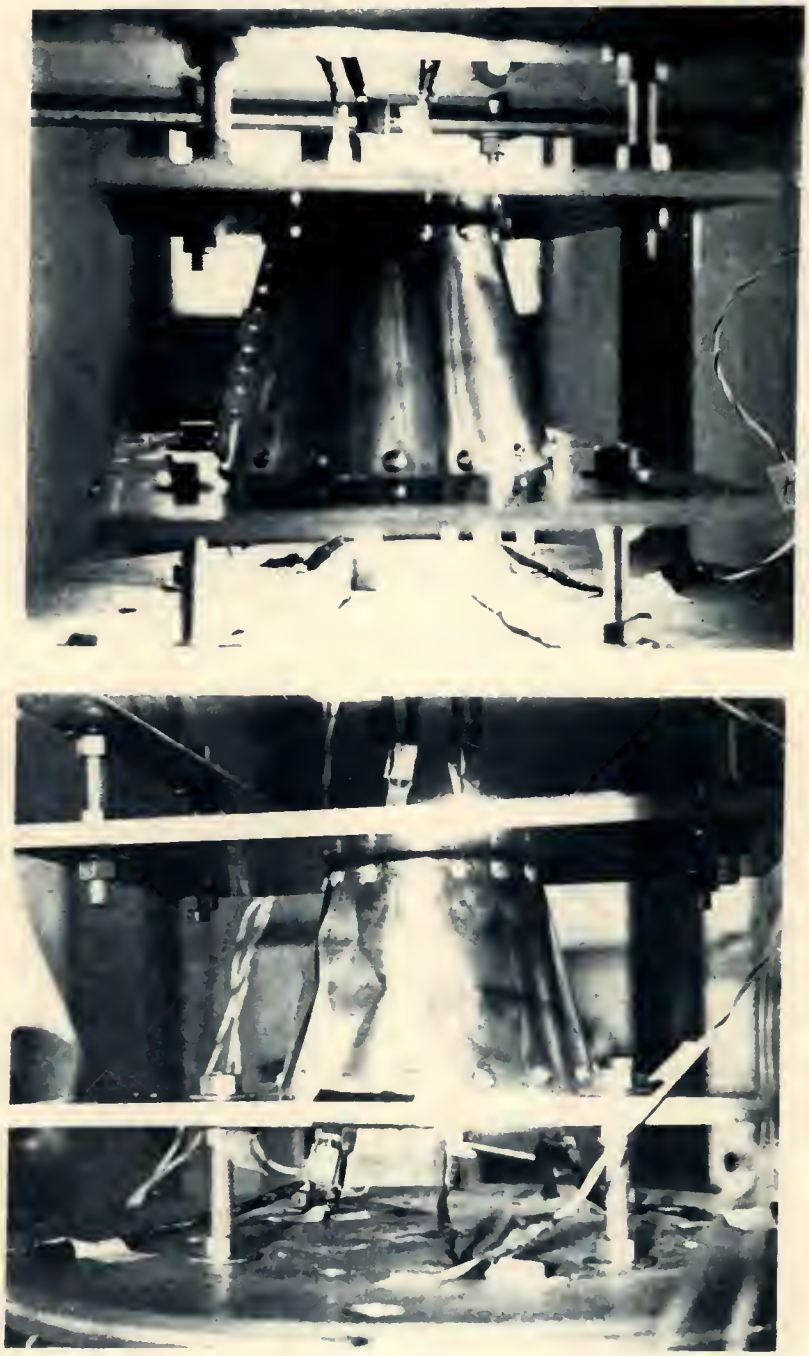


Fig. 15. Comparison of critical temperature with minimum temperature ($\beta = 10^{o}$, H/R = 2).

The dimension of the conical shells used in the test are: $\frac{R}{h} = 500$, $\frac{H}{R} = 2$ and $\beta = 15^{o}$ as shown in Fig. 2. The brass conical shells were mounted so that they were prevented from lengthening by two rigid plates, which were held in place by four 1/2-inch screw rods. At the two ends of the conical shell, two rings were made and mounted on the plates. The cone was then fixed on the rings by screws. Heat was provided by infra-red lamps, which were placed inside the shell and were designed for uniform temperature distribution over the surface of the shell. Thermal papers and thermal couples were both used to measure the temperature.

Three cones were made and tested; the buckling temperatures were found to be $120^{o}F$, $130^{o}F$ and $135^{o}F$, which are higher than the theoretical value ($98^{o}F$). This is primarily because the buckling was determined visually.

It is therefore reasonable to believe that the buckling occurs before it can be observed; in other words, the actual buckling temperature should be lower than that found experimentally. As temperature increases after buckling, the deformation increases, too. Fig. 16(A) shows the conical shell beginning to buckle, while Fig. 16(B) shows the buckling pattern of the cone as the temperature continues to increase after buckling.



(A)

(B)

Fig. 16. Photograph showing buckling of a heated conical shell

APPENDIX A

EQUATIONS FOR CASE I

$$\psi_i' = m_i'^2 + 1 - \lambda^2 , \qquad \Omega_i' = m_i'^4 + 2(1-3\lambda^2)m_i'^2 + (\lambda^2-1)^2$$

$$c_i' = [(m_i'^2 + \lambda - \lambda^2)\Omega_i' - 4(\lambda - 2\lambda^2)m_i'^2\psi_i](\Omega_i'^2 + 16\, m_i'^2\lambda^2\psi_i^2)^{-1}$$

$$d_i' = m_i'[(1-2\lambda)\Omega_i' + 4\lambda(m_i'^2 + \lambda - \lambda^2)\psi_i'](\Omega_i'^2 + 16\, m_i'^2\lambda^2\psi_i'^2)^{-1}$$

$$A_1 = (2 - 2\nu + 3g - \nu g + g^2)(2+g)^{-2}$$

$$A_2 = (2 - 2\nu + g - 3\nu g - \nu g^2)(2+g)^{-2}(1+g)^{-1}$$

$$A_3 = 6\lambda^2 - 12\lambda + 4 , \qquad A_4 = 4\lambda^3 - 12\lambda^2 + 8\lambda$$

$$A_5 = \lambda^4 - 4\lambda^3 + 4\lambda^2$$

$$A_6 = (1+\lambda)^2 d_1' + 2(1+\lambda)m_1'c_1' - m_1'^2 d_1'$$

$$A_7 = (1+\lambda)d_1' + m_1'c_1'$$

$$A_8 = (1+\lambda)^2 c_1' - 2(1+\lambda)m_1'd_1' - m_1'^2 c_1'$$

$$A_9 = (1+\lambda)c_1' - m_1'd_1'$$

$$A_{10} = (1+\lambda)^2 \alpha_3' + 2(1+\lambda) m_3' c_3' - m_3'^2 \alpha_3'$$

$$A_{11} = (1+\lambda) \alpha_3' + m_3' c_3'$$

$$A_{12} = (1+\lambda)^2 c_3' - 2(1+\lambda) m_3' \alpha_3' - m_3' c_3'$$

$$A_{13} = (1+\lambda) c_3' - m_3' \alpha_3'$$

$$A_{14} = (1+\lambda)^2 \alpha_5' + 2(1+\lambda) m_5' c_5' - m_5' \alpha_5'$$

$$A_{15} = (1+\lambda) \alpha_5' + m_5' c_5'$$

$$A_{16} = (1+\lambda)^2 c_5' - 2(1+\lambda) m_5' \alpha_5' - m_5' c_5'$$

$$A_{17} = (1+\lambda) c_5' - m_5' \alpha_5' \quad , \quad A_{18} = e^{2(\lambda-1)x_0} - 1$$

$$A_{19} = e^{2\lambda x_0} - 1 \quad , \quad A_{20} = e^{(3\lambda-1)x_0} - 1$$

$$A_{21} = (m_3'^4 - A_3 m_3'^2 + A_5)(1-\lambda) A_{18} \{ [4(\lambda-1)^2 + (m_1' + m_3')^2]^{-1} - [4(\lambda-1)^2 + (m_1' - m_3')^2]^{-1} \}$$

$$A_{22} = -0.5 A_{18} [4(1-\lambda) m_3'^4 + A_4 m_3'^2] \{ (m_1' + m_3') [4(1+\lambda)^2 + (m_1' + m_3')^2]^{-1} - (m_3' - m_1') [4(\lambda-1)^2 + (m_3' - m_1')^2] \}$$

$$A_{23} = \lambda D^{-1} A_{19} (A_{12} - A_{13}) \{ [4\lambda^2 + (m_1' + m_3')^2]^{-1} - [4\lambda^2 + (m_1' - m_3')^2]^{-1} \}$$

$$A_{24} = 0.5\, D^{-1} A_{19} (A_{10} - A_{11}) \{ (m_1' + m_3') [4\lambda^2 + (m_1' + m_3')^2]^{-1} - (m_3' - m_1') [4\lambda^2 + (m_3' - m_1')^2]^{-1} \}$$

$$A_{25} = A_{18} (m_3'^4 - A_3 m_3'^2 + A_5) \{ [4(\lambda - 1)]^{-1} - (\lambda - 1) [4(\lambda - 1)^2 + 4 m_3'^2]^{-1} \}$$

$$A_{26} = -A_{18} [4(1 - \lambda) m_3'^4 + A_4 m_3'^2] [4(\lambda - 1)^2 + 4 m_3'^2]^{-1}$$

$$A_{27} = A_{19} D^{-1} (A_{13} - A_{12}) [(4\lambda)^{-1} - \lambda (4\lambda^2 + 4 m_3'^2)^{-1}]$$

$$A_{28} = A_{19} D^{-1} m_3' (A_{10} - A_{11}) (4\lambda^2 + 4 m_3'^2)^{-1}$$

$$A_{29} = (m_5'^4 - A_3 m_5'^2 + A_5)(1 - \lambda) A_{18} \{ [4(\lambda - 1)^2 + (m_1' + m_5')^2]^{-1} - [4(\lambda - 1)^2 + (m_1' - m_5')^2]^{-1} \}$$

$$A_{30} = 0.5\, A_{18} [4(1 - \lambda) m_5'^4 + A_4 m_5'^2] \{ (m_1' + m_5') [(\lambda - 1)^2 + (m_1' + m_5')^2]^{-1} - (m_5' - m_1') [4(\lambda - 1)^2 + (m_3' - m_1')^2]^{-1} \}$$

$$A_{31} = \lambda D^{-1} A_{19} (A_{16} - A_{17}) \{ [4\lambda^2 + (m_1' + m_5')]^{-1} - [4\lambda^2 + (m_1' - m_5')^2]^{-1} \}$$

$$A_{32} = 0.5 D^{-1} A_{19} (A_{14} - A_{15}) \{ (m_1' + m_5') [4\lambda^2 + (m_1' + m_5')^2]^{-1} - (m_5' - m_1') [4\lambda^2 + (m_5' - m_1')^2]^{-1} \}$$

$$A_{33} = A_{18} (1-\lambda) (m_5'^4 - A_3 m_5'^2 + A_5) \{ [4(\lambda-1)^2 + (m_3' + m_5')^2]^{-1} - [4(\lambda-1)^2 + (m_3' - m_5')^2]^{-1} \}$$

$$A_{34} = -0.5 A_{18} [4(1-\lambda) m_5'^4 + A_4 m_5'^2] \{ (m_5' + m_3') [4(\lambda-1)^2 + (m_5' + m_3')^2]^{-1} + (m_3' - m_5') [4(\lambda-1)^2 + (m_3' - m_5')^2]^{-1} \}$$

$$A_{35} = \lambda D^{-1} A_{19} (A_{16} - A_{17}) \{ [4\lambda^2 + (m_5' + m_3')^2]^{-1} - [4\lambda^2 + (m_5' - m_3')^2]^{-1} \}$$

$$A_{36} = 0.5 D^{-1} A_{19} (A_{14} - A_{15}) \{ (m_3' + m_5') [4\lambda^2 + (m_3' + m_5')^2]^{-1} - (m_5' - m_3') [4(\lambda-1)^2 + (m_3' - m_5')^2]^{-1} \}$$

$$A_{37} = (1-\lambda) A_{18} (m_3'^4 - A_3 m_3'^2 + A_5) \{ [4(\lambda-1)^2 + (m_5' + m_3')^2]^{-1} - [4(\lambda-1)^2 + (m_5' - m_3')^2]^{-1} \}$$

$$A_{38} = -0.5 A_{18} [4(1-\lambda) m_3'^4 + A_4 m_3'^2] \{(m_3' + m_5')[4(\lambda-1)^2 + (m_5' + m_3')^2]^{-1} - (m_3' - m_5')[4(\lambda-1)^2 + (m_3' - m_5')^2]^{-1}\}$$

$$A_{39} = \lambda D^{-1} A_{19} (A_{12} - A_{13}) \{[4\lambda^2 + (m_5' + m_3')^2]^{-1} - [4\lambda^2 + (m_5' - m_3')^2]^{-1}\}$$

$$A_{40} = 0.5 D^{-1} A_{19} (A_{10} - A_{11}) \{(m_5' + m_3')[4\lambda^2 + (m_5' + m_3')^2]^{-1} - (m_3' - m_5')[4\lambda^2 + (m_3' - m_5')^2]^{-1}\}$$

$$A_{41} = A_{18} (m_5'^4 - A_3 m_5'^2 + A_5) \{[4(\lambda-1)]^{-1} - (\lambda-1)[4(\lambda-1)^2 + 4m_5'^2]^{-1}\}$$

$$A_{42} = -A_{18} [4(1-\lambda) m_5'^4 + A_4 m_5'^2][4(\lambda-1)^2 + 4m_5'^2]^{-1}$$

$$A_{43} = D^{-1} A_{19} (A_{17} - A_{16}) [(4\lambda)^{-1} - \lambda (4\lambda^2 + 4m_5'^2)^{-1}]$$

$$A_{44} = -D^{-1} A_{19} (A_{15} - A_{14}) (4\lambda^2 + 4m_5'^2)^{-1}$$

$$A_{45} = m_1' \{D[(1+\lambda+g)^2 + m_1'^2]\}^{-1} [1 + e^{(1+\lambda+g)x_0}]$$

$$A_{46} = m_3' \{D[(1+\lambda+g)^2 + m_3'^2]\}^{-1} [1 + e^{(1+\lambda+g)x_0}]$$

$$A_{47} = m_5' \{D[(1+\lambda+g)^2 + m_5'^2]\}^{-1} [1 + e^{(1+\lambda+g)x_0}]$$

$$A_{48} = -(2+3g+g^2)(g+2)^{-2}, \qquad A_{49} = -(2+g)^{-2}$$

$$C_1 = (1+e^{\lambda x_0})(\lambda^2+m_1'^2)^{-1}\{m_1'[A_9(1+\nu)-\nu A_8]+\lambda[\nu A_6 - (1+\nu)A_7]\}$$

$$C_2 = C_1^{-1}(1+\nu)(1-e^{-x_0}), \qquad C_3 = C_1^{-1}(2-2\nu)(e^{x_0}-1)$$

$$C_4 = [(1-3\nu)(e^{x_0}-1)+2(1-\nu)(1+x_0e^{x_0}-e^{x_0})]C_1^{-1}$$

$$C_5 = C_1^{-1}(1-e^{x_0}), \quad C_6 = [A_6-(1+\nu)A_7]^{-1}$$

$$C_7 = -C_6(1+\nu), \quad C_8 = 2C_6(1-\nu), \quad C_9 = C_6(3-\nu)$$

$$C_{10} = -C_7 e^{-(1+\lambda)x_0}, \qquad C_{11} = -C_8 e^{(1-\lambda)x_0}$$

$$C_{12} = -(C_9+C_8x_0)e^{(1-\lambda)x_0}, \qquad C_{13} = C_6 e^{(1-\lambda)x_0}$$

$$C_{14} = \{1+\gamma_1[(1+g)^{-1}-A_2][1-e^{(1+g)x_0}](1-e^{x_0})^{-1}\}C_5$$

$$C_{15} = [1+\gamma_1(1-A_1)]C_6, \qquad C_{16} = [1+(1-A_1)\gamma_1 e^{gx_0}]C_{13}$$

$$C_{17} = (1+e^{\lambda x_0})(\lambda^2+m_3'^2)^{-1}\{m_3'[(1+\nu)A_{13}-\nu A_{12}]+\lambda[\nu A_{10} - (1+\nu)A_{11}]\}C_1^{-1}$$

$$C_{18} = C_6[A_{10}-(1+\nu)A_{15}], \qquad C_{19} = C_6[A_{14}-(1+\nu)A_{15}]$$

$$C_{20} = (1+e^{\lambda x_0})(\lambda^2+m_5'^2)\{m_5'[(1+\nu)A_{17} - \nu A_{16}] + \lambda[\nu A_{14} - (1+\nu)A_{15}]\} C_1^{-1}$$

$$D_1 = (C_{14} - C_{15})(C_2 - C_7)^{-1}, \quad D_2 = (C_2 - C_8)(C_2 - C_7)^{-1}$$

$$D_3 = (C_4 - C_9)(C_2 - C_7)^{-1}, \quad D_4 = (C_{14} - C_{16})(C_2 - C_{10})^{-1}$$

$$D_5 = (C_3 - C_{11})(C_2 - C_{10})^{-1}, \quad D_6 = (C_4 - C_{12})(C_2 - C_{10})^{-1}$$

$$D_7 = (D_1 - D_4)(D_2 - D_5)^{-1}, \quad D_8 = (D_6 - D_3)(D_2 - D_5)^{-1}$$

$$D_9 = D_2 - D_1 D_7, \quad D_{10} = -D_2 D_8 - D_3, \quad D_{11} = (-C_2 D_{10} + C_3 D_8 + C_4)^{-1}$$

$$D_{12} = D_{11}(C_2 D_9 + C_3 D_7 - C_{14}), \quad D_{13} = D_{10} D_{11}, \quad D_{14} = D_9 + D_{10} D_{12}$$

$$D_{15} = D_8 D_{11}, \quad D_{16} = (C_{17} - C_{18})(C_2 - C_7)^{-1}, \quad D_{17} = (C_{17} - C_{18})(C_2 - C_{10})^{-1}$$

$$D_{18} = (D_{17} - D_{16})(D_2 - D_5)^{-1}, \quad D_{19} = -D_2 D_{18} - D_{16}$$

$$D_{20} = D_{11}(C_{17} + C_2 D_{19} + C_3 D_{18}), \quad D_{21} = D_{19} + D_{10} D_{20}$$

$$D_{22} = D_{18} + D_8 D_{20}, \quad D_{23} = (C_{20} - C_{19})(C_2 - C_7)^{-1}$$

$$D_{24} = (C_{20} - C_{19})(C_2 - C_{10})^{-1}, \quad D_{25} = (D_{24} - D_{23})(D_2 - D_5)^{-1}$$

$$D_{26} = -D_2 D_{25} - D_{23}, \qquad D_{27} = D_{11}(C_{20} + C_2 D_{26} + C_3 D_{25})$$

$$D_{28} = D_{26} + D_{10} D_{27}, \quad D_{29} = D_{25} + D_8 D_{27}, \quad D_{30} = D_7 + D_8 D_{12}$$

$$E_1 = A_{18}(m_1'^4 - A_3 m_1'^2 + A_5)\{[4(\lambda-1)]^{-1} - (\lambda-1)[4(\lambda-1)^2 + 4m_1'^2]^{-1}\}$$

$$E_2 = -A_{18}[4(1-\lambda)m_1'^4 + A_4 m_1'^2][4(\lambda-1)^2 + 4m_1'^2]^{-1}$$

$$E_3 = A_{19} D^{-1}(A_9 - A_8)[(4\lambda)^{-1} - \lambda(4\lambda^2 + 4m_1'^2)^{-1}]$$

$$E_4 = D^{-1} A_{19}(A_6 - A_7)(4\lambda^2 + 4m_1'^2)^{-1}$$

$$E_5 = D^{-1} m_1'[e^{(\lambda-1)x_0} + 1][(\lambda-1)^2 + m_1'^2]^{-1}$$

$$E_6 = -D^{-1} m_1'[e^{(\lambda+1)x_0} + 1][(\lambda+1)^2 + m_1'^2]^{-1}$$

$$E_7 = -2D^{-1}(1+\lambda)^2[(1+\lambda)^2 + m_1'^2]^{-1}\{-m_1'(1+\lambda)[e^{(1+\lambda)x_0} + 1][(1+\lambda)^2 + m_1'^2]^{-1} + m_1'(1+\lambda)^{-1}[x_0(1+\lambda)^{-1} - (1+\lambda)^{-2}]e^{(1+\lambda)x_0} - m_1'(1+\lambda)^{-3} + m_1'^3(1+\lambda)^{-3}[1 + e^{(1+\lambda)x_0}][(1+\lambda)^2 + m_1'^2]^{-1}\}$$

$$E_8 = E_5 D_{14} + 2E_6 D_{30} + 3E_6 D_{12} + E_7 D_{12}, \quad E_9 = E_1 + E_2$$

$$E_{10} = E_7 D_{11} + E_3 + E_4 + E_5 D_{13} + 2e_6 D_{15} + 3E_6 D_{11}$$

$$E_{11} = A_{18}(1-\lambda)(m_1'^4 - A_3 m_1'^2 + A_5)\{[4(\lambda-1)^2+(m_1'+m_3')^2]^{-1}$$

$$-[4(\lambda-1)^2+(m_1'-m_3')^2]^{-1}\}$$

$$E_{12} = -0.5\,A_{18}[4(1-\lambda)m_1'^4 + A_4 m_1'^2]\{(m_1'+m_3')[4(\lambda-1)^2+$$

$$(m_1'+m_3')^2]^{-1} + (m_3'-m_1')[4(\lambda-1)^2+(m_3'-m_1')^2]^{-1}\}$$

$$E_{13} = \lambda D^{-1} A_{19}\{[4\lambda^2+(m_1'+m_3')^2]^{-1} - [4\lambda^2+(m_3'-m_1')^2]^{-1}\}$$

$$E_{14} = 0.5\,A_{19} D^{-1}(A_6-A_7)\{(m_1'+m_3')[4\lambda^2+(m_1'+m_3')^2]^{-1} +$$

$$(m_3'-m_1')[4\lambda^2+(m_3'-m_1')^2]^{-1}\}$$

$$E_{15} = D^{-1} m_3'[e^{(\lambda-1)x_0}+1][(\lambda-1)^2+m_3'^2]^{-1}$$

$$E_{16} = -m_3' D^{-1}[e^{(1+\lambda)x_0}+1][(\lambda+1)^2+m_3'^2]^{-1}$$

$$E_{17} = -2D^{-1}(1+\lambda)^2[(1+\lambda)^2+m_3'^2]^{-1}\{-m_3'(1+\lambda)[e^{(1+\lambda)x_0}+1][(1+\lambda)^2$$

$$+m_1'^2]^{-1} + m_1'(1+\lambda)^{-1}[x_0(1+\lambda)^{-1}-(1+\lambda)^{-2}]e^{(1+\lambda)x_0} -$$

$$m_3'(1+\lambda)^{-3} + m_3'^3(1+\lambda)^{-3}[1+e^{(1+\lambda)x_0}][(1+\lambda)^2+m_3'^2]^{-1}\}$$

$$E_{18} = A_{21}+A_{22}, \quad E_{19} = A_{23}+A_{24}+D_{21}E_5+2E_6D_{22}+D_{20}(3E_6+E_7)$$

$$E_{20} = E_{11} + E_{12}, \quad E_{21} = E_{13} + E_{14} + E_{15} D_{13} + 2E_{16} D_{15} + D_{11} (3E_{16} + E_{17})$$

$$E_{22} = A_{25} + A_{26}, \quad E_{23} = A_{27} + A_{28} + E_{15} D_{21} + 2 E_{16} D_{22} + D_{20}(3E_{16} + E_{17})$$

$$E_{24} = E_{15} D_{14} + 2E_{16} D_{16} + D_{12}(3E_{16} + E_{17}), \quad E_{25} = E_9 + E_{10} \frac{\gamma_1^2}{h^2} \cot^2\beta$$

$$E_{26} = -E_{25}^{-1} (E_{18} + E_{19} \frac{\gamma_1^2}{h^2} \cot^2\beta), \quad E_{27} = -E_{25}^{-1} \frac{\gamma_1}{h} \cot\beta \, (E_8 - \gamma_1 A_{44} A_{41})$$

$$E_{28} = E_{20} + \frac{\gamma_1^2}{h^2} \cot^2\beta \; E_{21}, \qquad E_{29} = -E_{28}^{-1} (E_{22} + E_{23} \frac{\gamma_1^2}{h^2} \cot^2\beta)$$

$$E_{30} = -E_{28}^{-1} (E_{24} - \gamma_1 A_{44} A_{42}) \frac{\gamma_1}{h} \cot\beta$$

$$E_{31} = (E_{27} - E_{30})(E_{29} - E_{26})^{-1}, \quad E_{32} = E_{26} E_{31} + E_{27}$$

$$E_{33} = A_{18} (1-\lambda)(m_1'^4 - A_3 m_1'^2 + A_5) \left\{ [4(\lambda-1)^2 + (m_1' + m_3')^2]^{-1} - [4(\lambda-1)^2 + (m_1' - m_3')^2]^{-1} \right\}$$

$$E_{34} = 0.5 A_{18} [4(1-\lambda) m_1'^4 + A_4 m_1'^2] \left\{ (m_1' - m_5') [4(\lambda-1)^2 + (m_1' - m_5')^2]^{-1} - (m_5' + m_1') [4(\lambda-1)^2 + (m_5' + m_1')^2]^{-1} \right\}$$

$$E_{35} = \lambda D^{-1} (A_8 - A_9) A_{19} \left\{ [4\lambda^2 + (m_1' + m_5')^2]^{-1} - [4\lambda^2 + (m_5' - m_1')^2]^{-1} \right\}$$

$$E_{36} = 0.5 D^{-1} (A_6 - A_7) \left\{ (m_1' + m_5') [4\lambda^2 + (m_1' + m_5')^2]^{-1} - (m_1' - m_5') [4\lambda^2 + (m_1' - m_5')^2]^{-1} \right\}$$

$$E_{37} = m_5' D^{-1}\left[e^{(\lambda-1)x_0} + 1\right]\left[(\lambda-1)^2 + m_5'^2\right]^{-1}$$

$$E_{38} = -m_5' D^{-1}\left[e^{(\lambda+1)x_0} + 1\right]\left[(\lambda+1)^2 + m_5'^2\right]^{-1}$$

$$E_{39} = -2D^{-1}(1+\lambda)^2\left[(1+\lambda)^2 + m_5'^2\right]^{-1}\Big\{-m_5'(1+\lambda)\left[e^{(1+\lambda)x_0} + 1\right] x$$

$$\left[(1+\lambda)^2 + m_5'^2\right]^{-1} + m_5'(1+\lambda)^{-1}\left[x_0(1+\lambda)^{-1} - (1+\lambda)^{-2}\right]e^{(1+\lambda)x_0}$$

$$- m_5'(1+\lambda)^{-3} + m_5'(1+\lambda)^{-3}\left[1 + e^{(1+\lambda)x_0}\right]\left[(1+\lambda)^2 + m_5'^2\right]^{-1}\Big\}$$

$$E_{40} = A_{29} + A_{30}, \qquad E_{41} = A_{31} + A_{32} + E_5 D_{28} + 2E_6 D_{29} + D_{27}(3E_6 + E_7)$$

$$E_{42} = A_{33} + A_{34}, \qquad E_{43} = A_{35} + A_{36} + E_{15} D_{28} + 2E_{16} D_{29} + D_{27}(3E_6 + E_7)$$

$$E_{44} = E_{33} + E_{34}, \qquad E_{45} = E_{35} + E_{36} + E_{37} D_{13} + 2E_{38} D_{15} + D_{11}(3E_{38} + E_{39})$$

$$E_{46} = A_{37} + A_{38}, \qquad E_{47} = A_{39} + A_{40} + E_{37} D_{21} + 2E_{38} D_{22} + D_{20}(3E_{38} + E_{39})$$

$$E_{48} = A_{41} + A_{42}, \qquad E_{49} = A_{43} + A_{44} + E_{37} D_{28} + 2E_{38} D_{29} + D_{27}(3E_{38} + E_{39})$$

$$E_{50} = E_{37} D_{14} + 2E_{38} D_{16} + (3E_{38} + E_{39}) D_{12} - A_{48} A_{47} \frac{\eta_1}{h} \cot\beta$$

$$E_{51} = -E_{25}^{-1}\left(E_{40} + E_{41} \frac{\eta_1^2}{h^2} \cot^2\beta\right), \qquad E_{52} = -E_{28}^{-1}\left(E_{42} + E_{43} \frac{\eta_1^2}{h^2} \cot^2\beta\right)$$

$$E_{53} = (E_{51} - E_{52})(E_{29} - E_{26})^{-1}, \qquad E_{54} = E_{44} + E_{45} \frac{\eta_1^2}{h^2} \cot^2\beta$$

$$E_{55} = -E_{54}^{-1}\left(E_{46} + E_{47}\frac{r_1^2}{h^2}\cot^2\beta\right), \quad E_{56} = -E_{54}^{-1}\left(E_{48} + E_{49}\frac{r_1^2}{h^2}\cot^2\beta\right)$$

$$E_{57} = E_{50}E_{54}^{-1}\frac{r_1}{h}\cot\beta, \qquad E_{58} = (E_{56} - E_{52})(E_{29} - E_{55})^{-1}$$

$$E_{59} = (E_{57} - E_{30})(E_{29} - E_{55})^{-1}, \quad E_{60} = (E_{31} - E_{59})(E_{58} - E_{53})^{-1}$$

$$E_{61} = E_{58}E_{60} + E_{59}, \qquad E_{62} = E_{26}E_{61} + E_{51}E_{60} + E_{27}$$

$$a_1' = E_{62}\left(\frac{r_1^2}{h^2}\alpha T_1\right), \quad a_3' = E_{61}\left(\frac{r_1^2}{h^2}\alpha T_1\right), \quad a_5' = E_{60}\left(\frac{r_1^2}{h^2}\alpha T_1\right)$$

$$A_1' = \left(\frac{r_1^2}{h^2}\alpha T_1\right)\left[D_{14} + \frac{r_1}{h}\cot\beta\left(E_{61}D_{21} + E_{62}D_{13} + E_{60}D_{28}\right)\right]$$

$$A_2' = \left(\frac{r_1^2}{h^2}\alpha T_1\right)\left[D_{30} + \frac{r_1}{h}\cot\beta\left(E_{61}D_{22} + E_{62}D_{15} + E_{60}D_{29}\right)\right]$$

$$A_3' = \left(\frac{r_1^2}{h^2}\alpha T_1\right)\left[D_{12} + \frac{r_1}{h}\cot\beta\left(E_{61}D_{20} + E_{62}D_{11} + E_{60}D_{27}\right)\right]$$

$$\psi_i'' = m_i''^2 + n^2 + 1 - \lambda^2, \qquad \Omega_i'' = (m_i''^2 + n^2)^2 - 2(1+\lambda^2)n^2 + 2(1-3\lambda^2)m_i''^2$$

$$C_i'' = \left[(m_i''^2 + \lambda - \lambda^2)\Omega_i'' - 4(\lambda - 2\lambda^2)m_i''\psi_i''\right]\left[\Omega_i''^2 + 16m_i''^2\lambda^2\psi_i''^2\right]^{-1}$$

$$\alpha_i'' = m_i''\left[(1-2\lambda)\Omega_i'' + 4\lambda(m_i''^2 + \lambda - \lambda^2)\psi_i''\right]\left[\Omega_i''^2 + 16m_i''^2\lambda^2\psi_i''^2\right]^{-1}$$

$$A_{50} = \lambda^4 - 6m_1''^2 + m_1''^4 - 4\lambda^3 + 12m_1''^2\lambda + 4\lambda^2 - 4m_1''^2 - n^2 + 2\lambda n^2(2-\lambda) + n^2(2m_1''^2 + n^2)$$

$$A_{51} = 4\lambda^3 m_1'' - 4m_1''^3\lambda - 12\lambda^2 m_1'' + 4m_1''^3 + 8m_1''\lambda + 4m_1''n^2 - 4m_1''n^2\lambda$$

$$A_{52} = \lambda^4 - 6m_3''^2\lambda^2 + m_3''^4 - 4\lambda^3 + 12m_3''^2\lambda + 4\lambda^2 - 4m_3''^2 - n^2 + 2n^2\lambda(2-\lambda) + n^2(2m_3''^2 + n^2)$$

$$A_{53} = 4\lambda^3 m_3'' - 4m_3''^3\lambda - 12\lambda^2 m_3'' + 4m_3''^3 + 8m_3''\lambda + 4m_3'' n^2 - 4m_3'' n^2\lambda$$

$$A_{54} = \lambda(1+\lambda)c_1' - (1+2\lambda)m_1'\alpha_1' - m_1'^2 c_1', \quad A_{55} = (1+\lambda)c_1' - m_1'\alpha_1'$$

$$A_{56} = \lambda(1+\lambda)\alpha_1' + (1+2\lambda)m_1'c_1' - m_1'^2\alpha_1', \quad A_{57} = (1+\lambda)\alpha_1' + m_1'c_1'$$

$$A_{58} = \lambda(1+\lambda)c_3' - (1+2\lambda)m_3'\alpha_3' - m_3'c_3', \quad A_{59} = (1+\lambda)c_3' - m_3'\alpha_3'$$

$$A_{60} = \lambda(1+\lambda)\alpha_3' + (1+2\lambda)m_3'c_3' - m_3'^2\alpha_3', \quad A_{61} = (1+\lambda)\alpha_3' + m_3'c_3'$$

$$A_{62} = \lambda(1+\lambda)c_5' - (1+2\lambda)m_5'\alpha_5' - m_5'c_5', \quad A_{63} = (1+\lambda)c_5' - m_5'\alpha_5$$

$$A_{64} = \lambda(1+\lambda)\alpha_5' + (1+2\lambda)m_5'c_5' - m_5'^2\alpha_5', \quad A_{65} = (1+\lambda)\alpha_5' + m_5'c_5'$$

$$A_{66} = (1+\lambda)^2\alpha_1'' + 2(1+\lambda)m_1''c_1'' - m_1''^2\alpha_1'', \quad A_{67} = (1+\lambda)\alpha_1'' + m_1''c_1''$$

$$A_{68} = (1+\lambda)^2 c_1'' - 2(1+\lambda)m_1''\alpha_1'' - m_1''^2 c_1'', \quad A_{69} = (1+\lambda)c_1'' - m_1''\alpha_1''$$

$$A_{70} = (1+\lambda)^2\alpha_3'' + 2(1+\lambda)m_3''c_3'' - m_3''^2\alpha_3'', \quad A_{71} = (1+\lambda)\alpha_3'' + m_3''c_3''$$

$$A_{72} = (1+\lambda)^2 c_3'' - 2(1+\lambda)m_3''\alpha_3'' - m_3''c_3'', \quad A_{73} = (1+\lambda)c_3'' - m_3''\alpha_3''$$

$$A_{74} = 1 + e^{(3\lambda-1)x_0}, \qquad A_{75} = e^{(2\lambda+g)x_0} - 1$$

$$Q_1 = 0.25 A_{74}\{2m_i'[(3\lambda-1)^2+m_i'^2]^{-1} - (2m_i''+m_i')[(3\lambda-1)^2+(2m_i''+m_i')^2]^{-1}$$

$$+ (2m_i''-m_i')[(3\lambda-1)^2+(2m_i''-m_i')^2]^{-1}\}$$

$$Q_2 = 0.25 A_{74}(3\lambda-1)\{-[(3\lambda-1)^2+(2m_i''-m_i')^2]^{-1}+[(3\lambda-1)^2+(2m_i''+m_i')^2]^{-1}\}$$

$$Q_3 = 0.25 A_{74}(3\lambda-1)\{-2[(3\lambda-1)^2+m_i'^2]^{-1}+[(3\lambda-1)^2+(2m_i''+m_i')^2]^{-1}$$

$$+ [(3\lambda-1)^2+(2m_i''-m_i')^2]^{-1}\}$$

$$Q_4 = 0.25 A_{74}\{(2m_i''+m_i')[(3\lambda-1)^2+(2m_i''+m_i')^2]^{-1}+(2m_i''-m_i')[(3\lambda-1)^2+(2m_i''-m_i')^2]^{-1}\}$$

$$Q_5 = 0.25 m_i''^2(\lambda-1)^{-1}A_{18}[(1-\lambda)^2+m_i''^2]^{-1}$$

$$Q_6 = -m_i'' A_{18}[4(1-\lambda)^2+4m_i''^2]^{-1}, \qquad Q_7 = A_{19} m_i''^2 \lambda^{-1}(4\lambda^2+4m_i'')^{-1}$$

$$Q_8 = -0.25 m_i'' A_{19}(\lambda^2+m_i''^2)^{-1}$$

$$Q_9 = 0.125(2x_0-\lambda^{-2})e^{2\lambda x_0}+0.125\lambda^{-2}-\lambda x_0 e^{2\lambda x_0}(4\lambda^2+4m_i''^2)^{-1}$$

$$- A_{19}(0.125\lambda^2-0.125m_i''^2)(\lambda^2+m_i''^2)^{-2}$$

$$Q_{10} = 0.25[-x_0 m_i'' e^{2\lambda x_0}(\lambda^2+m_i''^2)^{-1}+m_i'' A_{19}(\lambda^2+m_i''^2)^{-2}]$$

$$Q_{11} = 0.25 A_{18}\{(\lambda-1)^{-1}-(\lambda-1)[(\lambda-1)^2+m_i''^2]^{-1}\}$$

$$Q_{12} = -0.25\, m_1''\left[(\lambda-1)^2 + m_1''^2\right]^{-1}$$

$$Q_{13} = A_{18}\left\{\left[4(\lambda-1)^2 + (m_3'' - m_1'')^2\right]^{-1} - \left[4(\lambda-1)^2 + (m_3'' + m_1'')^2\right]^{-1}\right\}$$

$$Q_{14} = 0.5A_{18}\left\{-(m_3'' + m_1'')\left[4(\lambda-1)^2 + (m_3'' + m_1')^2\right]^{-1} - (m_1'' - m_3'')\left[4(\lambda-1)^2 + (m_3'' - m_1')^2\right]^{-1}\right\}$$

$$Q_{15} = 0.25A_{14}\left\{2m_3'\left[(3\lambda-1)^2 + m_3'^2\right]^{-1} - (2m_1'' + m_3')\left[(3\lambda-1)^2 + (2m_1'' + m_3')^2\right]^{-1} + (2m_1'' - m_3')\left[(3\lambda-1)^2 + (2m_1'' - m_3')^2\right]^{-1}\right\}$$

$$Q_{16} = -0.5A_{18}\left\{(m_1'' + m_3'')\left[4(\lambda-1)^2 + (m_1'' + m_3'')^2\right]^{-1} + (m_3'' - m_1'')\left[4(\lambda-1)^2 + (m_3'' - m_1'')^2\right]^{-1}\right\}$$

$$Q_{17} = 0.25A_{18}\left\{(\lambda-1)^{-1} - \left[(\lambda-1)^2 + m_3''^2\right]^{-1}\right\}$$

$$Q_{18} = -0.25 m_3'' A_{18}\left[(\lambda-1)^2 + m_3''^2\right]^{-1}$$

$$Q_{19} = 0.5A_{19}\left\{0.5\lambda^{-1} - \lambda\left[2\lambda^2 + 2m_1''^2\right]^{-1}\right\}$$

$$Q_{20} = -m_1'' A_{19}\left[4\lambda^2 + 4m_1''^2\right]^{-1}$$

$$Q_{21} = A_{19}\left\{\left[4\lambda^2 + (m_3'' - m_1'')^2\right]^{-1} - \left[4\lambda^2 + (m_3'' + m_1'')^2\right]^{-1}\right\}$$

$$Q_{22} = -0.5A_{19}\left\{(m_3''+m_1'')\left[4\lambda^2+(m_3''+m_1'')^2\right]^{-1}+(m_1''-m_3'')\left[4\lambda^2+(m_1''-m_3'')^2\right]^{-1}\right\}$$

$$Q_{23} = 0.25A_{74}(3\lambda-1)\left\{-\left[(3\lambda-1)^2+(2m_1''-m_3')^2\right]^{-1}+\left[(3\lambda-1)^2+(2m_1''+m_3'^2)\right]^{-1}\right\}$$

$$Q_{24} = 0.5A_{19}\left\{(m_3''+m_1'')\left[4\lambda^2+(m_3''+m_1'')^2\right]^{-1}+(m_3''-m_1'')\left[4\lambda^2+(m_3''-m_1'')^2\right]^{-1}\right\}$$

$$Q_{25} = 0.25A_{19}\left[\lambda^{-1}-\lambda\left(\lambda^2+m_3''^2\right)^{-1}\right]$$

$$Q_{26} = -0.25A_{19}m_3''\left(m_3''^2+\lambda^2\right)^{-1}$$

$$Q_{27} = 0.25A_{74}(3\lambda-1)\left\{-2\left[(3\lambda-1)^2+m_3'^2\right]^{-1}+\left[(3\lambda-1)^2+(2m_1''+m_3')^2\right]^{-1}\right.$$
$$\left.+\left[(3\lambda-1)^2+(2m_1''-m_3')^2\right]^{-1}\right\}$$

$$Q_{28} = 0.25A_{74}\left\{(2m_1''+m_3')\left[(3\lambda-1)^2+(2m_1''+m_3')^2\right]^{-1}+(2m_1''-m_3')\times\right.$$
$$\left.\left[(3\lambda-1)^2+(2m_1''-m_3')^2\right]^{-1}\right\}$$

$$Q_{29} = 0.5\left[4\lambda^2+(m_3''-m_1'')^2\right]^{-1}\left\{2\lambda x_0 e^{2\lambda x_0}-\left[4\lambda^2-(m_3''-m_1'')^2\right]\left[4\lambda^2+\right.\right.$$
$$\left.\left.(m_3''-m_1'')^2\right]^{-1}A_{19}\right\}-0.5\left[4\lambda^2+(m_3''+m_1'')^2\right]^{-1}\left\{2\lambda x_0 e^{2\lambda x_0}-\right.$$
$$\left.\left[4\lambda^2-(m_3''+m_1'')^2\right]\left[4\lambda^2+(m_3''+m_1'')^2\right]^{-1}A_{19}\right\}$$

$$Q_{30} = 0.25A_{74}\left\{(m_1'+m_3''+m_1'')\left[(3\lambda-1)^2+(m_1'+m_3''+m_1')^2\right]^{-1}+\right.$$

$$(m_1'-m_3''+m_1'')\left[(3\lambda-1)^2+(m_1'-m_3''+m_1'')^2\right]^{-1}-(m_1'+m_3''+m_1'')\left[(3\lambda$$

$$-1)^2+(m_1'+m_3''+m_1'')^2\right]^{-1}-(m_1'-m_3''-m_1'')\left[(3\lambda-1)^2+(m_1'-m_3''-m_1'')^2\right]^{-1}\Big\}$$

$$Q_{31}=0.25A_{74}(3\lambda-1)\Big\{\left[(3\lambda-1)^2+(m_1'+m_3''+m_1'')^2\right]^{-1}+\left[(3\lambda-1)^2+(m_1'-m_3''\right.$$

$$\left.-m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_1'+m_3''-m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_1'-m_3''+m_1'')^2\right]^{-1}\Big\}$$

$$Q_{32}=0.25A_{74}\Big\{(m_3'+m_3''-m_1'')\left[(3\lambda-1)^2+(m_3'+m_3''-m_1'')^2\right]^{-1}+(m_3'-m_3''$$

$$+m_1'')\left[(3\lambda-1)^2+(m_3'-m_3''+m_1'')^2\right]^{-1}-(m_3'+m_3''+m_1'')\left[(3\lambda-1)^2\right.$$

$$\left.+(m_3'+m_3''+m_1'')^2\right]^{-1}-(m_3'-m_3''-m_1'')\left[(3\lambda-1)^2+(m_3'-m_3''-m_1'')^2\right]^{-1}\Big\}$$

$$Q_{33}=0.25A_{74}(3\lambda-1)\Big\{\left[(3\lambda-1)^2+(m_3'+m_3''+m_1'')^2\right]^{-1}+\left[(3\lambda-1)^2+(m_3'-m_3''\right.$$

$$\left.-m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_3'+m_3''-m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_3'-m_3''+m_1'')^2\right]^{-1}\Big\}$$

$$Q_{34}=A_{74}(3\lambda-1)\Big\{\left[(3\lambda-1)^2+(m_3'+m_3''-m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_3'-m_3''+m_1'')^2\right]^{-1}$$

$$+\left[(3\lambda-1)^2+(m_3'+m_3''+m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_3'-m_3''-m_1'')^2\right]^{-1}\Big\}$$

$$Q_{35}=-0.25A_{74}\Big\{(m_3'-m_1''-m_3'')\left[(3\lambda-1)^2+(m_3'-m_1''-m_3'')^2\right]^{-1}-(m_3'+m_1''+m_3'')\times$$

$$\left[(3\lambda-1)^2+(m_3'+m_1''+m_3'')^2\right]^{-1}+(m_1'+m_3''-m_1'')\left[(3\lambda-1)^2+(m_1'+m_3''-m_1'')^2\right]^{-1}-$$

$$(m_3'+m_1''-m_3'')\left[(3\lambda-1)^2+(m_3'+m_1'-m_3'')^2\right]^{-1}\Big\}$$

$$Q_{36} = 0.5\,(m_1''+m_3'')\left[4\lambda^2+(m_1''+m_3'')^2\right]^{-1}\Big\{-x_0 e^{2\lambda x_0}+4\lambda A_{19}\left[4\lambda^2+(m_1''+m_3'')^2\right]^{-1}+0.5\,(m_1''-m_3'')\left[4\lambda^2+(m_3''-m_1'')^2\right]^{-1}\Big\}-x_0 e^{2\lambda x_0}+4\lambda A_{19}\left[4\lambda^2+(m_3''-m_1'')^2\right]^{-1}\Big\}$$

$$Q_{37} = A_{74}(3\lambda-1)\Big\{\left[(3\lambda-1)^2+(m_1'+m_3''-m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_1'-m_3''+m_1'')^2\right]^{-1}+\left[(3\lambda-1)^2+(m_1'+m_3''+m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_1'-m_3''-m_1'')^2\right]^{-1}\Big\}(0.25)$$

$$Q_{38} = -0.25A_{74}\Big\{(m_1'-m_1''-m_3'')\left[(3\lambda-1)^2+(m_1'-m_1''-m_3'')^2\right]^{-1}-(m_1'+m_3''+m_1'')\times\left[(3\lambda-1)^2+(m_1''+m_3''+m_1'')^2\right]^{-1}+(m_1'-m_1''+m_3'')\left[(3\lambda-1)^2+(m_1'-m_1''+m_3'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_1'+m_1''-m_3'')^2\right]^{-1}(m_1'+m_1''-m_3'')\Big\}$$

$$Q_{39} = 0.25A_{74}(3\lambda-1)\Big\{\left[(3\lambda-1)^2+(m_3'+m_1''+m_3'')^2\right]^{-1}+\left[(3\lambda-1)^2+(m_3'+m_3''+m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_3'-m_3''-m_1'')^2\right]^{-1}-\left[(3\lambda-1)^2+(m_3'-m_3''+m_1'')^2\right]^{-1}\Big\}$$

$$Q_{40} = 0.25A_{74}\Big\{(m_3''+m_1''+m_3')\left[(3\lambda-1)^2+(m_3''+m_1''+m_3')^2\right]^{-1}+(m_3''+m_1''-m_3')\times\left[(3\lambda-1)^2+(m_3''+m_1''-m_3')^2\right]^{-1}+(m_3''-m_1''+m_3')\left[(3\lambda-1)^2+(m_3''-m_1''+m_3')^2\right]^{-1}$$

$$-(m_3''-m_1''-m_3')[(3\lambda-1)^2+(m_3''-m_1''-m_3')^2]^{-1}\}$$

$$Q_{41}=0.25A_{74}\{2m_3'[(3\lambda-1)^2+m_3'^2]^{-1}-(2m_3''+m_3')[(3\lambda-1)^2+(2m_3''+m_3')^2]^{-1}$$

$$+(2m_3''-m_3')[(3\lambda-1)^2+(2m_3''-m_3')^2]^{-1}\}$$

$$Q_{42}=0.25A_{74}\{-2[(3\lambda-1)^2+m_3'^2]^{-1}+[(3\lambda-1)^2+(2m_3''+m_3')^2]^{-1}+[(3\lambda-1)^2$$

$$+(2m_3''-m_3')^2]^{-1}\}$$

$$Q_{43}=0.25(3\lambda-1)A_{74}\{-[(3\lambda-1)^2+(2m_3''-m_3')^2]^{-1}+[(3\lambda-1)^2+(2m_3''+m_3'^2)]^{-1}\}$$

$$Q_{44}=-0.5A_{18}\{(m_3''+m_1'')[4(\lambda-1)^2+(m_3''+m_1'')^2]+(m_3''-m_1'')[4(\lambda-1)^2+(m_3''-m_1'')^2]^{-1}\}$$

$$Q_{45}=-0.5A_{19}\{(m_3''+m_1'')[4\lambda^2+(m_3''+m_1'')^2]^{-1}+(m_3''-m_1'')[4\lambda^2+(m_3''-m_1'')^2]^{-1}\}$$

$$Q_{46}=0.5(m_3''+m_1'')[4\lambda^2+(m_3''+m_1'')^2]^{-1}\{-x_0e^{2\lambda x_0}+4\lambda A_{19}[4\lambda^2+(m_3''+m_1'')^2]^{-1}\}$$

$$-0.5(m_1''-m_3'')[4\lambda^2+(m_1''-m_3'')^2]^{-1}\{-x_0e^{2\lambda x_0}+4\lambda A_{19}[4\lambda^2+(m_1''-m_3'')^2]^{-1}\}$$

$$Q_{47}=0.25A_{74}(3\lambda-1)\{[(3\lambda-1)^2+(m_1'+m_1''+m_3'')^2]^{-1}+[(3\lambda-1)^2+(m_1'+m_3''-$$

$$m_1'')^2]^{-1}-[(3\lambda-1)^2+(m_1'-m_3''-m_1'')^2]^{-1}-[(3\lambda-1)^2+(m_1'-m_3''-m_1'')^2]^{-1}\}$$

$$Q_{48}=0.25A_{74}\{(m_3''+m_1''+m_1')[(3\lambda-1)^2+(m_3''+m_1''+m_1')^2]^{-1}+(m_3''+m_1''-m_1')[(3\lambda-1)^2+$$

$$(m_3''+m_1''-m_1')^2]^{-1}+(m_3''-m_1''+m_1')[(3\lambda-1)^2+(m_3''-m_1''+m_1')^2]^{-1}-$$

$$(m_3''-m_1''-m_1'')[(3\lambda-1)^2+(m_3''-m_1''-m_1')^2]^{-1}\}$$

$$Q_{49}=0.25A_{18}(\lambda-1)^{-1}m_3''^2[(1-\lambda)^2+m_3''^2]^{-1}$$

$$Q_{50}=-0.25A_{19}\lambda^{-1}m_3''^2(\lambda^2+m_3''^2)^{-1}$$

$$Q_{51}=0.5e^{2\lambda x_0}(0.5x_0\lambda^{-1}-0.25\lambda^{-2})+0.125\lambda^{-2}-\lambda x_0 e^{2\lambda x_0}(4\lambda^2+4m_3''^2)^{-1}$$

$$+0.125A_{19}(\lambda^2-m_3''^2)(\lambda^2+m_3''^2)^{-2}$$

$$Q_{52}=0.25A_{74}\{2m_1'[(3\lambda-1)^2+m_1'^2]^{-1}-(2m_3''+m_1')[(3\lambda-1)^2+(2m_3''+m_1')^2]^{-1}$$

$$+(2m_3''-m_1')[(3\lambda-1)^2+(2m_3''-m_1')^2]^{-1}\}$$

$$Q_{53}=0.25A_{74}\{-2[(3\lambda-1)^2+m_1'^2]^{-1}+[(3\lambda-1)^2+(2m_3''+m_1')^2]^{-1}+[(3\lambda-1)^2$$

$$+(2m_3''-m_1')^2]^{-1}\}$$

$$Q_{54}=-0.25m_3''A_{18}[(1-\lambda)^2+m_3''^2]^{-1},\qquad Q_{55}=-0.25A_{19}m_3''(\lambda^2+m_3''^2)^{-1}$$

$$Q_{56}=0.25[m_3''x_0e^{2\lambda x_0}(\lambda^2+m_3''^2)^{-1}+m_3''\lambda A_{19}(\lambda^2+m_3''^2)^{-2}]$$

$$Q_{57}=0.25A_{74}(3\lambda-1)\{-[(3\lambda-1)^2+(2m_3''-m_1')^2]^{-1}+[(3\lambda-1)^2+(2m_3''+m_1')^2]^{-1}\}$$

$$Q_{58} = 0.25A_{74}\{(2m_3'' + m_1')[(3\lambda-1)^2 + (2m_3'' + m_1')^2]^{-1} + (2m_3'' - m_1')[(3\lambda-1)^2 + (2m_3'' - m_1')^2]^{-1}\}$$

$$Q_{59} = 2A_{75}m_1''^2(2\lambda+g)^{-1}[(2\lambda+g)^2 + 4m_1''^2]^{-1}$$

$$Q_{60} = -m_1''A_{75}[(2\lambda+g)^2 + 4m_1''^2]^{-1}$$

$$Q_{61} = 0.25A_{75}(2\lambda+g)\{[(2\lambda+g)^2 + (m_3'' - m_1'')^2]^{-1} - [(2\lambda+g)^2 + (m_3'' + m_1'')^2]^{-1}\}$$

$$Q_{62} = -0.25A_{75}\{(m_3'' + m_1'')[(2\lambda+g)^2 + (m_3'' - m_1'')^2]^{-1} + (m_1'' - m_3'')[(2\lambda+g)^2 + (m_3'' - m_1'')^2]^{-1}\}$$

$$Q_{63} = 0.25A_{74}\{(m_3' + 2m_3'')[(3\lambda-1)^2 + (m_3' + 2m_3'')^2]^{-1} + (2m_3'' - m_3')[(3\lambda-1)^2 + (2m_3'' - m_3')^2]^{-1}\}$$

$$Q_{64} = -0.25A_{75}\{(m_3'' + m_1'')[(2\lambda+g)^2 + (m_3'' + m_1'')^2]^{-1} + (m_3'' - m_1'')\times[(2\lambda+g)^2 + (m_3'' - m_1'')^2]^{-1}\}$$

$$Q_{65} = 2A_{74}m_3''^2(2\lambda+g)^{-1}[(2\lambda+g)^2 + 4m_3''^2]^{-1}$$

$$Q_{66} = -A_{74}m_3''[(2\lambda+g)^2 + 4m_3''^2]^{-1}$$

$$Q_{67} = 0.25A_{74}\{2m_5'[(3\lambda-1)^2+m_5'^2]^{-1} - (2m_1''+m_5')[(3\lambda-1)^2+(2m_1''+m_5')^2]^{-1}$$

$$+ (2m_1''-m_5')[(3\lambda-1)^2+(2m_1''-m_5')^2]^{-1}\}$$

$$Q_{68} = 0.25A_{74}(3\lambda-1)\{-[(3\lambda-1)^2+(2m_1''-m_5')^2]^{-1}+[(3\lambda-1)^2+(2m_1''+m_5')^2]^{-1}\}$$

$$Q_{69} = 0.25A_{74}(3\lambda-1)\{-2[(3\lambda-1)^2+m_5'^2]^{-1}+[(3\lambda-1)^2+(2m_1''+m_5')^2]^{-1}+$$

$$[(3\lambda-1)^2+(2m_1''-m_5')^2]^{-1}\}$$

$$Q_{70} = 0.25A_{74}\{(2m_1''+m_5')[(3\lambda-1)^2+(2m_1''+m_5')^2]^{-1}+(2m_1''-m_5')[(3\lambda-1)^2$$

$$+(2m_1''-m_5')^2]^{-1}\}$$

$$Q_{71} = 0.25A_{74}\{(m_5'+m_3''-m_1'')[(3\lambda-1)^2+(m_5'+m_3''-m_1'')^2]^{-1}+(m_5'-m_3''+$$

$$m_1'')[(3\lambda-1)^2+(m_5'-m_3''+m_1'')^2]^{-1}-(m_5'+m_3''+m_1'')[(3\lambda-1)^2+$$

$$(m_5'+m_3''+m_1'')^2]^{-1}-(m_5'-m_3''-m_1'')[(3\lambda-1)^2+(m_5'-m_3''-m_1'')^2]^{-1}\}$$

$$Q_{72} = 0.25A_{74}(3\lambda-1)\{[(3\lambda-1)^2+(m_5'+m_3''+m_1'')^2]^{-1}+[(3\lambda-1)^2+(m_5'-$$

$$m_3''-m_1'')^2]^{-1}-[(3\lambda-1)^2+(m_5'+m_3''-m_1'')^2]-[(3\lambda-1)^2+(m_5'-m_3''+m_1'')^2]^{-1}\}$$

$$Q_{73} = 0.25A_{74}\{[(3\lambda-1)^2+(m_5'+m_3''-m_1'')^2]^{-1}-[(3\lambda-1)^2+(m_5'-m_3''+m_1'')^2]^{-1}+$$

$$[(3\lambda-1)^2+(m_5'+m_3''+m_1'')^2]^{-1}-[(3\lambda-1)^2+(m_5'-m_3''-m_1'')^2]^{-1}\}$$

$$Q_{74}=-0.25A_{74}\{(m_5'-m_1''-m_3'')[(3\lambda-1)^2+(m_5'-m_1''-m_3'')^2]^{-1}-$$

$$(m_5'+m_3''+m_1'')[(3\lambda-1)^2+(m_5'+m_3''+m_1'')^2]^{-1}+(m_5'-m_1''+m_3'')[(3\lambda-1)^2$$

$$+(m_5'-m_1''+m_3'')^2]^{-1}-(m_5'+m_1''-m_3'')[(3\lambda-1)^2+(m_5'+m_1''+m_3'')^2]^{-1}\}$$

$$Q_{75}=0.25A_{74}\{(m_3'+m_3''-m_1'')[(3\lambda-1)^2+(m_3'+m_3''-m_1'')^2]^{-1}+(m_3'-m_3''+m_1'')\times$$

$$[(3\lambda-1)^2+(m_3'-m_3''+m_1'')^2]^{-1}-(m_3'+m_3''+m_1'')[(3\lambda-1)^2+(m_3'+m_3''+$$

$$m_1'')^2]^{-1}-(m_3'-m_3''-m_1'')[(3\lambda-1)^2+(m_3'-m_3''-m_1'')^2]^{-1}\}$$

$$Q_{76}=0.25A_{74}(3\lambda-1)\{[(3\lambda-1)^2+(m_3'+m_3''+m_1'')^2]^{-1}+[(3\lambda-1)^2+(m_3'-$$

$$m_3''-m_1'')^2]^{-1}+[(3\lambda-1)^2+(m_3'+m_3''-m_1'')^2]^{-1}+[(3\lambda-1)^2+(m_3'-$$

$$m_3''+m_1'')^2]^{-1}\}$$

$$Q_{77}=0.25A_{74}(3\lambda-1)\{[(3\lambda-1)^2+(m_5'+m_1''+m_3'')^2]^{-1}+[(3\lambda-1)^2+(m_5'+$$

$$m_3''-m_1'')^2]^{-1}-[(3\lambda-1)^2+(m_5'-m_3''-m_1'')^2]^{-1}-[(3\lambda-1)^2+(m_5'-m_3''+m_1'')^2]^{-1}\}$$

$$Q_{78}=0.25A_{74}\{(m_3''+m_1''+m_5')[(3\lambda-1)^2+(m_3''+m_1''+m_5')^2]^{-1}+$$

$$(m_3''+m_1''-m_5')[(3\lambda-1)^2+(m_3''+m_1''-m_5')^2]^{-1}+(m_3''-m_1''+m_5')[(3\lambda-1)^2+$$

$$(m_3''-m_1''+m_5')^2]^{-1}-(m_3''-m_1''-m_5')[(3\lambda-1)^2+(m_3''-m_1''-m_5')^2]^{-1}\}$$

$$Q_{79}=0.25A_{74}\{2m_5'[(3\lambda-1)^2+m_5'^2]^{-1}-(2m_3''+m_5')[(3\lambda-1)^2+(2m_3''+$$

$$m_5')^2]^{-1}+(2m_3''-m_5')[(3\lambda-1)^2+(2m_3''-m_5')^2]^{-1}\}$$

$$Q_{80}=0.25A_{74}\{-2[(3\lambda-1)^2+m_5'^2]^{-1}+[(3\lambda-1)^2+(2m_3''+m_5')^2]^{-1}+$$

$$[(3\lambda-1)^2+(2m_3''-m_5')^2]^{-1}\}$$

$$Q_{81}=0.25A_{74}(3\lambda-1)\{-[(3\lambda-1)^2+(2m_3''-m_5')^2]+[(3\lambda-1)^2+(2m_3''+m_5')^2]^{-1}\}$$

$$Q_{82}=0.25A_{74}\{(2m_3''+m_5')[(3\lambda-1)^2+(2m_3''+m_5')^2]^{-1}+(2m_3''-m_5')\times$$

$$[(3\lambda-1)^2+(2m_3''-m_5')^2]^{-1}\}$$

$$X_1=A_{50}Q_{11}+A_{51}Q_{12}\,,\qquad X_2=A_{52}Q_{13}+A_{53}Q_{14}$$

$$X_3=A_{50}Q_{15}+A_{51}Q_{16}\,,\qquad X_4=A_{51}Q_{17}+A_{53}Q_{18}$$

$$X_5=(A_{66}-A_{68})Q_{19}+(A_{67}-A_{69})Q_{20}$$

$$X_6=(A_{70}-A_{72})Q_{21}+(A_{71}-A_{73})Q_{22}$$

$$X_7 = (A_{66} - A_{68})\, Q_{23} + (A_{67} - A_{69})\, Q_{24}$$

$$X_8 = (A_{70} - A_{72})\, Q_{25} + (A_{71} - A_{73})\, Q_{26}$$

$$X_9 = A_{48}\left[(\lambda - n^2)\, Q_{59} + m_1''\, Q_{60}\right], \quad X_{10} = A_{48}\left[(\lambda - n^2) Q_{61} + m_3''\, Q_{62}\right]$$

$$X_{11} = A_{48}\left[(\lambda - n^2) Q_{61} + m_1''\, Q_{64}\right], \quad X_{12} = A_{48}\left[(\lambda - n^2) Q_{65} + m_3''\, Q_{66}\right]$$

$$X_{13} = A_{49}\left[(\lambda^2 - \lambda - m_1''^2)\, Q_{59} + m_1''(2\lambda - 1)\, Q_{60}\right]$$

$$X_{14} = A_{49}\left[(\lambda^2 - \lambda - m_3''^2)\, Q_{61} + m_3''(2\lambda - 1)\, Q_{62}\right]$$

$$X_{15} = A_{49}\left[(\lambda^2 - \lambda - m_1''^2)\, Q_{61} + m_1''(2\lambda - 1)\, Q_{64}\right]$$

$$X_{16} = A_{49}\left[(\lambda^2 - \lambda - m_3^2)\, Q_{65} + m_3''(2\lambda - 1)\, Q_{66}\right]$$

$$Z_1 = (\lambda - n^2)\left[-A_1' Q_5 + (2A_2' + 3A_3')\, Q_7 + 2A_3' Q_9 + a_1' \frac{\gamma_1}{h} \cot\beta\, (A_{54} Q_1 \right.$$

$$+ A_{56} Q_3) + a_3' \frac{\gamma_1}{h} \cot\beta\, (A_{58} Q_{15} + A_{60} Q_{27}) + a_5' \frac{\gamma_1}{h} \cot\beta\, (A_{62} Q_{67}$$

$$\left. + A_{64} Q_{69})\right] + m_1''\left[-A_1' Q_6 + (2A_2' + 3A_3')\, Q_8 + 2A_3' Q_{10} + \right.$$

$$a_1' \frac{\gamma_1}{h} \cot\beta\, (A_{54} Q_2 + A_{56} Q_4) + a_3' \frac{\gamma_1}{h} \cot\beta\, (A_{58} Q_{23} + A_{60} Q_{28})$$

$$\left. + a_5' \frac{\gamma_1}{h} \cot\beta\, (A_{62} Q_{68} + A_{64} Q_{70})\right]$$

$$Z_2 = (\lambda - n^2)\left[-A_1' Q_{13} + (2A_2' + 3A_3') Q_{21} + 2A_3' Q_{29} + a_1' \frac{\gamma_1}{h} \cot\beta\,(A_{54} Q_{30}\right.$$

$$+ A_{56} Q_{31}) + a_3' \frac{\gamma_1}{h} \cot\beta\,(A_{58} Q_{32} + A_{60} Q_{33}) + a_5' \frac{\gamma_1}{h} \cot\beta\,(A_{62} Q_{71}$$

$$\left. + A_{64} Q_{72})\right] + m_3''\left[-A_1' Q_{14} + (2A_2' + 3A_3') Q_{22} + 2A_3' Q_{36} + \right.$$

$$a_1' \frac{\gamma_1}{h} \cot\beta\,(A_{54} Q_{37} + A_{56} Q_{39}) + a_3' \frac{\gamma_1}{h} \cot\beta\,(A_{58} Q_{34} + A_{60} Q_{35})$$

$$\left. + a_5' \frac{\gamma_1}{h} \cot\beta\,(A_{62} Q_{73} + A_{64} Q_{74})\right]$$

$$Z_3 = (\lambda - n^2)\left[-A_1' Q_{13} + (2A_2' + 3A_3') Q_{21} + 2A_3' Q_{29} + a_1' \frac{\gamma_1}{h} \cot\beta\,(A_{54} Q_{30}\right.$$

$$+ A_{56} Q_{31}) + a_3' \frac{\gamma_1}{h} \cot\beta\,(A_{58} Q_{75} + A_{60} Q_{76}) + a_5' \frac{\gamma_1}{h} \cot\beta\,(A_{62} Q_{71}$$

$$\left. + A_{64} Q_{72})\right] + m_1''\left[-A_1' Q_{44} + (2A_2' + 3A_3') Q_{45} + 2A_3' Q_{46} + a_1' \frac{\gamma_1}{h} \cot\beta\right.$$

$$\times (A_{54} Q_{47} + A_{56} Q_{48}) + a_3' \frac{\gamma_1}{h} \cot\beta\,(A_{58} Q_{39} + A_{60} Q_{40}) +$$

$$\left. a_5' \frac{\gamma_1}{h} \cot\beta\,(A_{62} Q_{77} + A_{64} Q_{78})\right]$$

$$Z_4 = (\lambda - n^2)\left[-A_1' Q_{13} + (2A_2' + 3A_3') Q_{50} + 3A_3' Q_{51} + a_1' \frac{\gamma_1}{h} \cot\beta\,(A_{54} Q_{52}\right.$$

$$+ A_{56} Q_{53}) + a_3' \frac{\gamma_1}{h} \cot\beta\,(A_{58} Q_{41} + A_{60} Q_{42}) + a_5' \frac{\gamma_1}{h} \cot\beta\,(A_{62} Q_{79}$$

$$\left. + A_{64} Q_{80})\right] + m_3''\left[-A_1' Q_{54} + (2A_2' + 3A_3') Q_{55} + 2A_3' Q_{56}\right.$$

$$+ a_1' \frac{\gamma_1}{h} \cot\beta \,(A_{54} Q_{57} + A_{56} Q_{58}) + a_3' \frac{\gamma_1}{h} \cot\beta \,(A_{58} Q_{43} +$$

$$A_{60} Q_{63}) + a_5' \frac{\gamma_1}{h} \cot\beta \,(A_{62} Q_{81} + A_{64} Q_{82})]$$

$$Z_5 = (\lambda^2 - \lambda - m_1''^2)[A_1' Q_5 + (2A_2' + A_3') Q_7 + 2A_3' Q_9 + a_1' \frac{\gamma_1}{h} \cot\beta \,(A_{55} Q_1 +$$

$$A_{57} Q_3) + a_3' \frac{\gamma_1}{h} \cot\beta \,(A_{59} Q_{15} + A_{61} Q_{27}) + a_5' \frac{\gamma_1}{h} \cot\beta \,(A_{63} Q_{67}$$

$$+ A_{65} Q_{69})] + m_1'' (2\lambda - 1)[A_1' Q_6 + (2A_2' + A_3') Q_8 + 2A_3' Q_{10} +$$

$$a_1' \frac{\gamma_1}{h} \cot\beta \,(A_{55} Q_2 + A_{57} Q_4) + a_3' \frac{\gamma_1}{h} \cot\beta \,(A_{59} Q_{23} + A_{61} Q_{28})$$

$$+ a_5' \frac{\gamma_1}{h} \cot\beta \,(A_{63} Q_{68} + A_{65} Q_{70})]$$

$$Z_6 = (\lambda^2 - \lambda - m_3''^2)[A_1' Q_{13} + (2A_2' + A_3') Q_{21} + 2A_3' Q_{29} + a_1' \frac{\gamma_1}{h} \cot\beta \,(A_{55} Q_{30}$$

$$+ A_{57} Q_{31}) + a_3' \frac{\gamma_1}{h} \cot\beta \,(A_{59} Q_{32} + A_{61} Q_{33}) + a_5' \frac{\gamma_1}{h} \cot\beta \,(A_{63} Q_{71}$$

$$+ A_{65} Q_{72})] + m_3'' (2\lambda - 1)[A_1' Q_{14} + (2A_2' + A_3') Q_{22} + 2A_3' Q_{36}$$

$$+ a_1' \frac{\gamma_1}{h} \cot\beta \,(A_{55} Q_{37} + A_{57} Q_{38}) + a_3' \frac{\gamma_1}{h} \cot\beta \,(A_{59} Q_{34} + A_{61} Q_{35})$$

$$+ a_5' \frac{\gamma_1}{h} \cot\beta \,(A_{63} Q_{73} + A_{65} Q_{74})]$$

$$Z_7 = (\lambda^2 - \lambda - m_1''^2)[A_1' Q_{13} + (2A_2' + A_3') Q_{21} + 2A_3' Q_{29} + a_1' \frac{\gamma_1}{h} \cot\beta \,(A_{55} Q_{30} + A_{57} Q_{31})$$

$$+ a_3' \frac{r_1}{h}\cot\beta(A_{59}Q_{75}+A_{61}Q_{76}) + a_5' \frac{r_1}{h}\cot\beta(A_{63}Q_{75}+A_{65}Q_{76})]$$

$$+ m_1''(2\lambda-1)[A_1'Q_{44}+(2A_2'+A_3')Q_{45}+2A_3'Q_{46}+a_1'\frac{r_1}{h}\cot\beta(A_{55}Q_{47}$$

$$+ A_{57}Q_{48}) + a_3'\frac{r_1}{h}\cot\beta(A_{59}Q_{39}+A_{61}Q_{40}) + a_5'\frac{r_1}{h}\cot\beta(A_{63}Q_{77}$$

$$+ A_{65}Q_{78})]$$

$$Z_8 = (\lambda^2-\lambda-m_3''^2)[A_1'Q_{49}+(2A_2'+A_3')Q_{50}+2A_3'Q_{51}+a_1'\frac{r_1}{h}\cot\beta(A_{55}Q_{52}$$

$$+ A_{57}Q_{52}) + a_3'\frac{r_1}{h}\cot\beta(A_{59}Q_{41}+A_{61}Q_{42}) + a_5'\frac{r_1}{h}\cot\beta(A_{63}Q_{79}$$

$$+ A_{65}Q_{80})] + m_3''(2\lambda-1)[A_1'Q_{54}+(2A_2'+A_3')Q_{55}+2A_3'Q_{56}$$

$$+ a_1'\frac{r_1}{h}\cot\beta(A_{55}Q_{57}+A_{57}Q_{58}) + a_3'\frac{r_1}{h}\cot\beta(A_{59}Q_{43}+A_{61}Q_{63})$$

$$+ a_5'\frac{r_1}{h}\cot\beta(A_{63}Q_{81}+A_{65}Q_{82})]$$

$$c_{11} = DX_1 - (\frac{r_1^2}{h^2}\cot^2\beta)X_5 \quad , \qquad d_{11} = -\frac{r_1^2}{h^2}(Z_1+Z_5)$$

$$f_{11} = -\frac{r_1^2}{h^2}(X_9+X_{13}) \quad , \qquad c_{13} = DX_2 - (\frac{r_1^2}{h^2}\cot^2\beta)X_6$$

$$d_{13} = -\frac{r_1^2}{h^2}(Z_2+Z_6) \quad , \qquad f_{13} = -\frac{r_1^2}{h^2}(X_{10}+X_{14})$$

$$c_{31} = DX_3 - (\frac{r_1^2}{h^2}\cot^2\beta)X_7, \qquad d_{31} = -\frac{r_1^2}{h^2}(Z_3+Z_7)$$

$$f_{31} = -\frac{r_i^2}{h^2}(X_{11} + X_{15}), \qquad c_{33} = DX_4 - \left(\frac{r_i^2}{h^2}\cot^2\beta\right) X_8$$

$$d_{33} = -\frac{r_i^2}{h^2}(Z_4 + Z_8), \qquad f_{33} = -\frac{r_i^2}{h^2}(X_{12} + X_{16})$$

$$Z_9 = C_{11}C_{33} - C_{13}C_{31}$$

$$Z_{10} = C_{33}(d_{11} + \gamma_i f_{11}) + C_{11}(d_{33} + \gamma_i f_{33}) - C_{13}(d_{31} + \gamma_i f_{33}) -$$

$$C_{31}(d_{13} + \gamma_i f_{31})$$

$$Z_{11} = (d_{11} + \gamma_i f_{11})(d_{33} + \gamma_i f_{33}) - (d_{13} + \gamma_i f_{13})(d_{31} + \gamma_i f_{31})$$

$$Z_{12} = Z_{10}^2 - 4 Z_9 Z_{11}, \qquad \alpha T_i = 0.5 Z_{11}^{-1}\left[-Z_{10} \pm (Z_{12})^{0.5}\right]$$

APPENDIX B

EQUATIONS FOR CASE II

$$g_i' = c_i' , \qquad h_i' = d_i' , \qquad R_{\alpha i} = (1+\lambda) g_i' + m_i' h_i'$$

$$R_{\alpha i}' = m_i' g_i' + (1+\lambda) h_i' , \qquad R_{\beta i} = \lambda R_{\alpha i} - m_i R_{\alpha i}' , \qquad R_{\beta i}' = m_i' R_{\alpha i} - \lambda R_{\alpha i}'$$

$$R_{\gamma i} = m_i'^4 + \lambda^4 - 6\lambda^2 m_i'^2 - 4\lambda^3 + 12\lambda m_i'^2 + 4\lambda^2 - 4 m_i'^2 , \qquad R_{\gamma i}' = 4\lambda m_i' (\lambda^2 - m_i'^2) + 4 m_i'^2 (3\lambda - 1)$$

$$i = 1, 3, 5$$

$$P_1 = -0.25 \gamma_2 x_0^{-1} (1 - e^{2x_0}) - (\pi x_0)^{-1} (1 - e^{x_0})$$

$$P_2 = 0.5 \pi^{-1} + \gamma_2 + 2P_1 - 0.5 \gamma_2 , \qquad P_3 = 0.25 x_0^{-1} \frac{\gamma_1}{h} \cot\beta \left[e^{(\lambda-1)x_0} + 1 \right] (R_{\alpha 1}' + R_{\beta 1}')$$

$$P_4 = 0.25 x_0^{-1} \frac{\gamma_1}{h} \cot\beta \left[e^{(\lambda-1)x_0} + 1 \right] (R_{\alpha 3}' + R_{\beta 3}')$$

$$P_5 = 0.25 x_0^{-1} \frac{\gamma_1}{h} \cot\beta \left[e^{(\lambda-1)x_0} + 1 \right] (R_{\alpha 5}' + R_{\beta 5}')$$

$$P_6 = -0.25 \frac{\gamma_1}{h} \cot\beta \, (R_{\alpha 1}' + R_{\beta 1}' + 4P_3) , \qquad P_7 = -0.25 \frac{\gamma_1}{h} \cot\beta \, (R_{\alpha 3}' + R_{\beta 3}' + 4P_4)$$

$$P_8 = -0.25 \frac{\gamma_1}{h} \cot\beta \, (R_{\alpha 5}' + R_{\beta 5}' + 4P_5)$$

$$P_9 = 0.5 x_0^{-1} \left\{ \left[1 - (1 + 2x_0) e^{2x_0} \right] P_3 + \frac{\gamma_1}{h} \cot\beta \left[1 + e^{(1+\lambda)x_0} \right] R_{\alpha 1}' \right\}$$

$$P_{10} = 0.5 x_0^{-1} \left\{ \left[1 - (1 + 2x_0) e^{2x_0} \right] P_4 + \frac{\gamma_1}{h} \cot\beta \left[1 + e^{(1+\lambda)x_0} \right] R_{\alpha 3}' \right\}$$

$$P_{11} = 0.5 x_0^{-1} \left\{ \left[1 - (1 + 2x_0) e^{2x_0} \right] P_5 + \frac{\gamma_1}{h} \cot\beta \left[1 + e^{(1+\lambda)x_0} \right] R_{\alpha 5}' \right.$$

$$P_{12} = \left(\frac{\gamma_1}{h} \cot\beta \right) R_{\beta 1}' + 2P_9 + 2P_6 + 3P_3$$

$$P_{13} = \left(\frac{\gamma_1}{h} \cot\beta \right) R_{\beta 3}' + 2P_{10} + 2P_7 + 3P_4$$

$$P_{14} = \left(\frac{\gamma_1}{h} \cot\beta \right) R_{\beta 5}' + 2P_{11} + 2P_8 + 3P_5$$

$$G_1 = 0.25 A_{18} (\lambda - 1)^{-1}, \quad G_2 = -A_{18} m_1' \left[4(\lambda-1)^2 + 4 m_1'^2 \right]^{-1}$$

$$G_3 = (\lambda - 1) A_{18} \left\{ \left[4(\lambda-1)^2 + (m_3' - m_1')^2 \right]^{-1} - \left[4(\lambda-1)^2 + (m_3' + m_1')^2 \right]^{-1} \right\}$$

$$G_4 = -0.5 A_{18} \left\{ (m_1' + m_3') \left[4(\lambda-1)^2 + (m_1' + m_3')^2 \right]^{-1} + (m_1' - m_3') \left[4(\lambda-1)^2 + (m_1' - m_3')^2 \right]^{-1} \right\}$$

$$G_5 = A_{18} (\lambda - 1) \left\{ \left[4(\lambda-1)^2 + (m_5' - m_1')^2 \right]^{-1} - \left[4(\lambda-1)^2 + (m_5' - m_1')^2 \right]^{-1} \right\}$$

$$G_6 = -0.5 A_{18} \left\{ (m_1' + m_5') \left[4(\lambda-1)^2 + (m_1' + m_5')^2 \right]^{-1} + (m_1' - m_5') \left[4(\lambda-1)^2 + (m_1' - m_5')^2 \right]^{-1} \right\}$$

$$G_7 = -0.5 A_{18} \left\{ (m_1' + m_3') \left[4(\lambda-1)^2 + (m_1' + m_3')^2 \right]^{-1} + (m_3' - m_1') \left[4(\lambda-1)^2 + (m_3' - m_1')^2 \right]^{-1} \right\}$$

$$G_8 = -A_{18} m_3' \left[4(\lambda-1)^2 + 4 m_3'^2 \right]^{-1}$$

$$G_9 = (\lambda - 1) A_{18} \left\{ \left[4(\lambda-1)^2 + (m_5' - m_3')^2 \right]^{-1} - \left[4(\lambda-1)^2 + (m_5' + m_3')^2 \right]^{-1} \right\}$$

$$G_{10} = -0.5 A_{18} \left\{ (m_5' + m_3') \left[4(\lambda-1)^2 + (m_5' + m_3')^2 \right]^{-1} + (m_5' - m_3') \left[4(\lambda-1)^2 + (m_5' - m_3')^2 \right]^{-1} \right\}$$

$$G_{11} = -0.5A_{18}\left\{(m_1'+m_5')\left[4(\lambda-1)^2+(m_1'+m_5')^2\right]^{-1}+(m_5'-m_1')\left[4(\lambda-1)^2+(m_5'-m_1')^2\right]^{-1}\right\}$$

$$G_{12} = -0.5A_{18}\left\{(m_3'+m_5')\left[4(\lambda-1)^2+(m_3'+m_5')^2\right]^{-1}+(m_5'-m_3')\left[4(\lambda-1)^2+(m_5'-m_3')^2\right]^{-1}\right\}$$

$$G_{13} = -A_{18}\,m_5'\left[4(\lambda-1)^2+4m_5'^2\right]^{-1}$$

$$G_{14} = 2m_1'^{-1}\left[1+e^{(\lambda-1)x_0}\right]-2(\lambda-1)^2m_1'^{-2}\left\{m_1'^{-1}(1-\lambda)^{-1}\left[1+e^{(\lambda-1)x_0}\right]+m_1'(\lambda-1)^{-1}\right.$$

$$\left.\times\left[x_0(\lambda-1)^{-1}-(\lambda-1)^{-2}\right]e^{(\lambda-1)x_0}-m_1'(\lambda-1)^{-3}+m_1'(\lambda-1)^{-3}\left[1+e^{(\lambda-1)x_0}\right]\right\}$$

$$G_{15} = -m_1'\left[1+e^{(\lambda-1)x_0}\right]\left[(\lambda-1)^2+m_1'^2\right]^{-1}$$

$$G_{16} = m_1'\left[1+e^{(1+\lambda)x_0}\right]\left[(\lambda+1)^2+m_1'^2\right]^{-1}$$

$$G_{17} = 3m_1'^{-1}\left[1+e^{(1+\lambda)x_0}\right]+2(1+\lambda)^2m_1'^{-2}\left\{(1+\lambda)m_1'^{-1}\left[1+e^{(\lambda-1)x_0}\right]+m_1'(1+\lambda)^{-1}\right.$$

$$\left.\times\left[x_0(1+\lambda)^{-1}-(\lambda+1)^{-2}\right]e^{(\lambda+1)x_0}-m_1'(\lambda+1)^{-3}+m_1'(1+\lambda)^{-3}\left[1+e^{(1+\lambda)x_0}\right]\right\}$$

$$G_{18} = 0.25A_{19}\lambda^{-1}\,, \qquad G_{19} = -A_{19}m_1'(4\lambda^2+4m_1'^2)^{-1}$$

$$G_{20} = \lambda A_{19}\left\{\left[4\lambda^2+(m_3'-m_1')^2\right]^{-1}-\left[4\lambda^2+(m_3'+m_1')^2\right]^{-1}\right\}$$

$$G_{21} = -0.25A_{19}\left\{(m_1'+m_3')\left[4\lambda^2+(m_1'+m_3')^2\right]^{-1}+(m_1'-m_3')\left[4\lambda^2+(m_1'-m_3')^2\right]^{-1}\right\}$$

$$G_{22} = \lambda A_{19}\left\{\left[4\lambda^2+(m_5'-m_1')^2\right]^{-1}-\left[4\lambda^2+(m_5'+m_1')^2\right]^{-1}\right\}$$

$$G_{23} = -0.5A_{19}\left\{(m_5'+m_1')\left[4\lambda^2+(m_5'+m_1')^2\right]^{-1}+(m_1'-m_5')\left[4\lambda^2+(m_1'-m_5')^2\right]^{-1}\right\}$$

$$G_{24} = 2m_3'^{-1}\left[1+e^{(\lambda-1)x_0}\right]-2(\lambda-1)^2 m_3'^{-2}\left\{m_3'^{-1}(1-\lambda)^{-1}\left[1+e^{(\lambda-1)x_0}\right]+m_3'(\lambda-1)^{-1}\right.$$

$$\left.\times\left[x_0(\lambda-1)^{-1}-(\lambda-1)^{-2}\right]e^{(\lambda-1)x_0}-m_3'(\lambda-1)^{-3}e^{(\lambda-1)x_0}\right\}$$

$$G_{25} = -\left[1+e^{(\lambda-1)x_0}\right]m_3'\left[(1-\lambda)^2+m_3'^2\right]^{-1}$$

$$G_{26} = m_3'\left[1+e^{(1+\lambda)x_0}\right]\left[(1+\lambda)^2+m_3'^2\right]^{-1}$$

$$G_{27} = 3m_3'^{-1}\left[1+e^{(\lambda+1)x_0}\right]-2(1+\lambda)m_3'^{-2}\left\{m_3'^{-1}(1+\lambda)\left[1+e^{(1+\lambda)x_0}\right]+m_3'(1+\lambda)^{-1}\right.$$

$$\left.\times\left[x_0(1+\lambda)^{-1}-(1+\lambda)^{-2}\right]e^{(1+\lambda)x_0}-m_3'(1+\lambda)^{-3}e^{(1+\lambda)x_0}\right\}$$

$$G_{28} = -0.5A_{19}\left\{(m_1'+m_3')\left[4\lambda^2+(m_1'+m_3')^2\right]^{-1}+(m_3'-m_1')\left[4\lambda^2+(m_3'-m_1')^2\right]^{-1}\right\}$$

$$G_{29} = -A_{19}m_3'\left(4\lambda^2+4m_3'^2\right)^{-1}$$

$$G_{30} = \lambda A_{19}\left\{\left[4\lambda^2+(m_5'-m_3')^2\right]^{-1}-\left[4\lambda^2+(m_5'+m_3')^2\right]^{-1}\right\}$$

$$G_{31} = -0.5A_{19}\left\{(m_5'+m_3')\left[4\lambda^2+(m_5'+m_3')^2\right]^{-1}+(m_3'-m_5')\left[4\lambda^2+(m_3'-m_5')^2\right]^{-1}\right\}$$

$$G_{32} = 2m_5'^{-1}\left[1+e^{(\lambda-1)x_0}\right]-2(\lambda-1)^2 m_5'^{-2}\left\{m_5'^{-1}(1-\lambda)^{-1}\left[1+e^{(\lambda-1)x_0}\right]+m_5'(\lambda-1)^{-1}\right.$$

$$\left.\times\left[x_0(\lambda-1)^{-1}-(\lambda-1)^{-2}\right]e^{(\lambda-1)x_0}-m_5'(\lambda-1)^{-3}e^{(\lambda-1)x_0}\right\}$$

$$G_{33} = -m_5' \left[1 + e^{(1+\lambda)x_0}\right]\left[(\lambda-1)^2 + m_5'^2\right]^{-1}$$

$$G_{34} = m_5' \left[1 + e^{(1+\lambda)x_0}\right]\left[(1+\lambda)^2 + m_5'^2\right]^{-1}$$

$$G_{35} = 3m_5' \left[1 + e^{(1+\lambda)x_0}\right] - 2(1+\lambda)m_5'^{-2}\left\{m_5'^{-1}(1+\lambda)\left[1 + e^{(1+\lambda)x_0}\right] + m_5'(1+\lambda)^{-1}\right.$$

$$\left. \times \left[x_0(1+\lambda)^{-1} - 2(1+\lambda)^{-2}\right] e^{(1+\lambda)x_0}\right\}$$

$$G_{36} = -0.5A_{19}\left\{(m_1' + m_5')\left[4\lambda^2 + (m_1' + m_5')^2\right]^{-1} + (m_5' - m_1')\left[4\lambda^2 + (m_5' - m_1')^2\right]^{-1}\right\}$$

$$G_{37} = -0.5A_{19}\left\{(m_3' + m_5')\left[4\lambda^2 + (m_3' + m_5')^2\right]^{-1} + (m_5' - m_3')\left[4\lambda^2 + (m_5' - m_3')^2\right]^{-1}\right\}$$

$$G_{38} = -A_{19}m_5'\left(4\lambda^2 + 4m_5'^2\right)^{-1}$$

$$G_{39} = D\left(G_1 R_{\delta 1} + G_2 R_{\delta 1}'\right) - \frac{\gamma_1^2}{h^2}\cot^2\beta\left(G_{18}R_{\beta 1} + G_{19}R_{\beta 1}'\right) - \frac{\gamma_1}{h}\cot\beta\left(G_{14}P_9\right.$$

$$\left. + G_{15}P_{12} + G_{16}P_6 + G_{17}P_3\right)$$

$$G_{40} = D\left(G_3 R_{\delta 3} + G_4 R_{\delta 3}'\right) - \frac{\gamma_1^2}{h^2}\cot^2\beta\left(G_{20}R_{\beta 3} + G_{21}R_{\beta 3}'\right) - \frac{\gamma_1}{h}\cot\beta\left(G_{14}P_{10}\right.$$

$$\left. + G_{15}P_{13} + G_{16}P_7 + G_{17}P_4\right)$$

$$G_{41} = D\left(G_5 R_{\delta 5} + G_6 R_{\delta 5}'\right) - \frac{\gamma_1^2}{h^2}\cot^2\beta\left(G_{22}R_{\beta 5} + G_{23}R_{\beta 5}'\right) - \frac{\gamma_1}{h}\cot\beta\left(G_{14}P_{11}\right.$$

$$\left. + G_{15}P_{14} + G_{16}P_8 + G_{17}P_5\right)$$

$$G_{42} = \frac{r_1}{h}\cot\beta\,(G_{14}P_{11} + G_{15}P_{14} + G_{16}P_8 + G_{17}P_5)$$

$$G_{43} = D(G_3 R_{\gamma 1} + G_7 R'_{\gamma 1}) - \frac{r_1^2}{h^2}\cot^2\beta\,(G_{20}R_{\beta 1} + G_{28}R'_{\beta 1}) - \frac{r_1}{h}\cot\beta\,(G_{24}P_9 + G_{25}P_{12} + G_{26}P_6 + G_{27}P_3)$$

$$G_{44} = D(G_1 R_{\gamma 3} + G_8 R'_{\gamma 3}) - \frac{r_1^2}{h^2}\cot^2\beta\,(G_{18}R_{\beta 3} + G_{29}R'_{\beta 3}) - \frac{r_1}{h}\cot\beta\,(G_{24}P_{10} + G_{25}P_{13} + G_{26}P_7 + G_{27}P_4)$$

$$G_{45} = D(G_9 R_{\gamma 5} + G_{10}R'_{\gamma 5}) - \frac{r_1^2}{h^2}\cot^2\beta\,(G_{30}R_{\beta 5} + G_{31}R'_{\beta 5}) - \frac{r_1}{h}\cot\beta\,(G_{24}P_{11} + G_{25}P_{14} + G_{26}P_8 + G_{27}P_5)$$

$$G_{46} = G_{24}P_1 + G_{25}P_2 - 0.25\gamma_2 G_{26}$$

$$G_{47} = D(G_5 R_{\gamma 1} + G_{11}R'_{\gamma 1}) - \frac{r_1^2}{h^2}\cot^2\beta\,(G_{22}R_{\beta 1} + G_{36}R'_{\beta 1}) - \frac{r_1}{h}\cot\beta\,(G_{32}P_9 + G_{33}P_{12} + G_{34}P_6 + G_{35}P_3)$$

$$G_{48} = D(G_8 R_{\gamma 3} + G_{12}R'_{\gamma 3}) - \frac{r_1^2}{h^2}\cot^2\beta\,(G_{30}R_{\beta 3} + G_3 R'_{\beta 3}) - \frac{r_1}{h}\cot\beta\,(G_{32}P_{10} + G_{33}P_{13} + G_{34}P_7 + G_{35}P_4)$$

$$G_{49} = D(G_1 R_{\gamma 3} + G_{13}R'_{\gamma 3}) - \frac{r_1^2}{h^2}\cot^2\beta\,(G_{18}R_{\beta 5} + G_{38}R'_{\beta 5}) - \frac{r_1}{h}\cot\beta\,(G_{32}P_{11}$$

$$+ G_{33} P_{14} + G_{34} P_8 + G_{35} P_5)$$

$$G_{50} = \frac{\gamma_1}{h} \cot\beta \, (G_{32} P_1 + G_{33} P_2 - 0.25 \gamma_2 G_{34})$$

$$G_{51} = [(G_{41} G_{47} - G_{49} G_{39})(G_{40} G_{43} - G_{44} G_{39}) - (G_{41} G_{43} - G_{45} G_{39})$$

$$\times (G_{40} G_{47} - G_{48} G_{39})][(G_{42} G_{47} - G_{39} G_{50})(G_{40} G_{43} - G_{44} G_{39})$$

$$- (G_{42} G_{43} - G_{46} G_{39})(G_{40} G_{47} - G_{48} G_{39})^{-1}]$$

$$G_{52} = (G_{40} G_{47} - G_{48} G_{39})^{-1} [(G_{42} G_{47} - G_{39} G_{50}) - G_{51} (G_{41} G_{47} - G_{49} G_{39})$$

$$G_{53} = G_{47}^{-1} (G_{50} - G_{48} G_{52} - G_{49} G_{51})$$

$$G_{54} = \frac{\gamma_1}{h} \cot\beta \left[e^{(\lambda-1)x_0} + 1 \right] \left[G_{53} (R'_{\alpha 1} + R'_{\beta 1}) + G_{52} (R'_{\alpha 3} + R'_{\beta 3}) + G_{51} (R'_{\alpha 5} + R'_{\beta 5}) \right]$$

$$G_{55} = 0.25 \left\{ -\gamma_2 - \frac{\gamma_1}{h} \cot\beta \left[G_{53} (R'_{\alpha 1} + R'_{\beta 1}) + G_{52} (R'_{\alpha 3} + R'_{\beta 3}) + G_{51} (R'_{\alpha 5} + R'_{\beta 5}) \right] - 4 G_{54} \right\}$$

$$G_{56} = P_1 + 0.5 x_0^{-1} \left[1 - (1 + 2x_0) e^{2x_0} \right] G_{54} + 0.5 x_0^{-1} \frac{\gamma_1}{h} \cot\beta \left[1 + e^{(1+\lambda)x_0} \right] (G_{53} R'_{\alpha 1}$$

$$+ G_{52} R'_{\alpha 3} + G_{51} R'_{\alpha 5})$$

$$G_{57} = \gamma_2 + 2\pi^{-1} + \frac{\gamma_1}{h} \cot\beta \, (G_{53} R'_{\beta 1} + G_{52} R'_{\beta 3} + G_{51} R'_{\beta 5}) + 2 G_{56} + 2 G_{55} + 3 G_{54}$$

$$b_1' = G_{53} \frac{\gamma_1^2}{h^2} \alpha \tau_2 , \qquad b_3' = G_{52} \frac{\gamma_1^2}{h^2} \alpha \tau_2 , \qquad b_5' = G_{51} \frac{\gamma_1^2}{h^2} \alpha \tau_2$$

$$B_1' = G_{56}\frac{r_1^2}{h^2}\alpha\tau_2, \quad B_2' = G_{57}\frac{r_1^2}{h_2}\alpha\tau_2, \quad B_3' = G_{55}\frac{r_1^2}{h^2}\alpha\tau_2, \quad B_4' = G_{54}\frac{r_1^2}{h^2}\alpha\tau_2$$

$$J_{\alpha i} = \lambda^4 - 6\lambda^2 m_i''^2 + m_i''^4 - 4\lambda^3 + 12 m_i''^2\lambda - 4 m_i''^2, \quad J_{\alpha i}' = -4 m_i''^4\lambda + 4\lambda^3 m_i'' - 12\lambda^2 m_i'' + 4 m_i''^3 + 8\lambda m_i''$$

$$J_{\beta i} = 0.5(n+k)^2\left[(n+k)^2 + 2m_i''^2 - 2\lambda^2 + 4(\lambda-1)\right], \quad J_{\beta i}' = 0.5(n-k)^2\left[-(n-k)^2 + 2(\lambda^2 - m_i''^2) - 4(\lambda-1)\right]$$

$$J_{\gamma i} = 2\lambda m_i''(n+k)^2(1-\lambda), \qquad J_{\gamma i}' = 2\lambda m_i''(n-k)^2(\lambda-1)$$

$$J_{\mu i} = (1+\lambda)^4 - 6(1+\lambda)^2 m_i''^2 + m_i''^4 - 4(1+\lambda)^3 + 12 m_i''^2(1+\lambda) - 4 m_i''^2 + 4(1+\lambda)^2 - 4(k+n)^2$$

$$+ 4(k+n)^2(1+\lambda) + 2m_i''(k+n)^2 - 2(1+\lambda)^2(k+n)^2 + (k+n)^4$$

$$J_{\mu i}' = 4m_i''\left[-(1+\lambda)^3 + m_i''^2(1+\lambda) + 3(1+\lambda)^2 - m_i''^2 - 2(1+\lambda) + (k+n)^2\lambda\right]$$

$$J_{\eta i} = -(1+\lambda)^4 + (1+\lambda)^2 m_i''^2 - m_i''^4 + 4(1+\lambda)^3 - 12 m_i''(1+\lambda) + 4 m_i''^2 - 4(1+\lambda)^2 + 4(k-n)^2$$

$$- 4(k-n)^2(1+\lambda) - 2m_i^2(k-n)^2 + 2(1+\lambda)^2(k-n)^2 - (k-n)^4$$

$$J_{\eta i}' = 4m_i''\left[(1+\lambda)^3 - m_i''^2(1+\lambda) - 3(1+\lambda)^2 + m_i''^2 + 2(1+\lambda) - \lambda(k-n)^2\right]$$

$$g_i'' = \left[(m_i''^2 + \lambda - \lambda^2)(J_{\mu i} - J_{\eta i}) - m_i(1-2\lambda)(J_{\mu i}' - J_{\eta i}')\right]\left[(J_{\mu i} - J_{\eta i})^2 + (J_{\mu i}' - J_{\eta i}')^2\right]^{-1}$$

$$h_i'' = \left[m_i''(1-2\lambda)(J_{\mu i} - J_{\eta i}) - (m_i''^2 + \lambda - \lambda^2)(J_{\eta i}' - J_{\mu i}')\right]\left[(J_{\mu i} - J_{\eta i})^2 + (J_{\mu i}' - J_{\eta i}')^2\right]^{-1}$$

$$i = 1, 3$$

$$Q_{83} = 0.125(\lambda-1)^{-1}\left[2x_0-(\lambda-1)^{-1}\right]+0.125(\lambda-1)^{-2}-(\lambda-1)x_0e^{2(\lambda-1)x_0}$$

$$\times\left[4(\lambda-1)^2+4m_1''^2\right]^{-1}-A_{19}\left[2(\lambda-1)^2-2m_1''^2\right]\left[4(\lambda-1)^2+4m_1''^2\right]^{-2}$$

$$Q_{84} = -m_1''x_0\left[4(\lambda-1)^2+4m_1''^2\right]^{-1}e^{2(\lambda-1)x_0}+4(\lambda-1)m_1''A_{19}\left[4(\lambda-1)^2+4m_1''^2\right]^{-1}$$

$$Q_{85} = 0.5\left[4(\lambda-1)^2+(m_3''-m_1'')^2\right]^{-1}\left\{2(\lambda-1)x_0e^{2(\lambda-1)x_0}-A_{19}\left[4(\lambda-1)^2-\right.\right.$$

$$\left.\left.(m_3''-m_1'')^2\right]\left[4(\lambda-1)^2+(m_3''-m_1'')^2\right]^{-1}\right\}-0.5\left[4(\lambda-1)^2+(m_3''+m_1'')^2\right]^{-1}$$

$$\times\left\{2(\lambda-1)x_0e^{2(\lambda-1)x_0}-A_{19}\left[4(\lambda-1)^2-(m_3''+m_1'')\right]\left[4(\lambda-1)^2+(m_3''+m_1'')^2\right]^{-1}\right\}$$

$$Q_{86} = 0.5(m_1''+m_3'')\left[4(\lambda-1)^2+(m_1''+m_3'')^2\right]^{-1}\left\{x_0e^{2(\lambda-1)x_0}+4(\lambda-1)A_{19}\left[4(\lambda-1)^2\right.\right.$$

$$\left.\left.+(m_1''+m_3'')^2\right]^{-1}\right\}+0.5(m_1''-m_3'')\left[4(\lambda-1)^2+(m_1''-m_3'')^2\right]^{-1}\left\{x_0e^{2(\lambda-1)x_0}\right.$$

$$\left.+4(\lambda-1)A_{19}\left[4(\lambda-1)^2+(m_1''-m_3'')^2\right]^{-1}\right\}$$

$$Q_{87} = 0.5(m_1''+m_3'')\left[4(\lambda-1)^2+(m_1''+m_3'')^2\right]^{-1}\left\{x_0e^{2(\lambda-1)x_0}+4(\lambda-1)A_{19}\left[4(\lambda-1)^2\right.\right.$$

$$\left.\left.+(m_1''+m_3'')^2\right]^{-1}\right\}-0.5(m_1''-m_3'')\left[4(\lambda-1)^2+(m_1''-m_3'')^2\right]^{-1}\left\{x_0e^{2(\lambda-1)x_0}\right.$$

$$\left.+4(\lambda-1)A_{19}\left[4(\lambda-1)^2+(m_1''-m_3'')^2\right]^{-1}\right\}$$

$$Q_{88} = 0.125(\lambda-1)^{-1}\left[2x_0-(\lambda-1)^{-1}\right]+0.125(\lambda-1)^{-2}-(\lambda-1)x_0e^{2(\lambda-1)x_0}$$

$$\times [4(\lambda-1)^2+4m_3''^2]^{-1} - A_{19}[2(\lambda-1)^2-2m_3''^2][4(\lambda-1)^2+4m_3''^2]^{-2}$$

$$Q_{89} = -m_3'' x_0 [4(\lambda-1)^2+4m_3''^2]^{-1} e^{2(\lambda-1)x_0} + 4(\lambda-1)m_3'' A_{19}[4(\lambda-1)^2+4m_3''^2]^{-1}$$

$$Q_{90} = 0.5[e^{(2\lambda-1)x_0}-1]\{(2\lambda-1)^{-1} - (2\lambda-1)[(2\lambda-1)^2+4m_1''^2]\}$$

$$Q_{91} = m_1''[1-e^{(2\lambda-1)x_0}][(2\lambda-1)^2+4m_1''^2]^{-1}$$

$$Q_{92} = [e^{(2\lambda-1)x_0}-1]\{[(2\lambda-1)^2+(m_3''-m_1'')^2]^{-1} - [(2\lambda-1)^2+(m_3''+m_1'')^2]^{-1}\}$$

$$Q_{93} = 0.5[1-e^{(2\lambda-1)x_0}]\{(m_3''+m_1'')[(2\lambda-1)^2+(m_3''+m_1'')^2]^{-1} + (m_1''-m_3'')$$

$$\times [(2\lambda-1)^2+(m_1''-m_3'')^2]^{-1}\}$$

$$Q_{94} = 0.5[1-e^{(2\lambda-1)x_0}]\{(m_1''+m_3'')[(2\lambda-1)^2+(m_1''+m_3'')^2]^{-1} + (m_3''-m_1'')$$

$$\times [(2\lambda-1)^2+(m_3''-m_1'')^2]^{-1}\}$$

$$Q_{95} = 0.5[e^{(2\lambda-1)x_0}-1]\{(2\lambda-1)^{-1} - (2\lambda-1)[(2\lambda-1)^2+4m_3''^2]\}$$

$$Q_{96} = [1-e^{(2\lambda-1)x_0}]m_3''[(2\lambda-1)^2+4m_3''^2]^{-1}$$

$$l_1 = 2(2\pi+k)^{-1} - (2\pi+3k)^{-1} + (2\pi-k)^{-1} + 8(3k)^{-1}$$

$$l_2 = (2\pi-k)^{-1} - (2\pi-3k)^{-1} - (2\pi+k)^{-1} - 8(3k)^{-1}$$

$$\ell_3 = -0.5\left[(n+k)^2 - \lambda\right], \qquad \ell_4 = 0.5\left[(n+k)^2 - \lambda\right]$$

$$I_{\alpha i} = -g_i'' m_i''^2 - 2(1+\lambda) m_i'' k_i'' + g_i''(1+\lambda)\lambda + k_i''^2 m_i''$$

$$I_{\alpha i}'' = k_i''(1+\lambda)\lambda + 2m_i''(1+\lambda) g_i'' - m_i''^2 k_i'' - g_i'' m_i''$$

$$i = 1, \ 3$$

$$I_{\beta i} = g_i'(1+\lambda) - k_i' m_i', \qquad I_{\beta' i} = g_i' m_i' + k_i'(1+\lambda)$$

$$i = 1, \ 3, \ 5$$

$$L_1 = Q_{11}(J_{\alpha 1} + J_{\beta 1} - J_{\beta' 1}) + Q_{12}(J_{\alpha' 1} + J_{\gamma 1} - J_{\gamma' 1})$$

$$L_2 = Q_{13}(J_{\alpha 3} + J_{\beta 3} - J_{\beta' 3}) + Q_{14}(J_{\alpha' 3} + J_{\gamma 3} - J_{\gamma' 3})$$

$$L_3 = Q_{13}(J_{\alpha 1} + J_{\beta 1} - J_{\beta' 1}) + Q_{16}(J_{\alpha' 1} + J_{\gamma 1} - J_{\gamma' 1})$$

$$L_4 = Q_{17}(J_{\alpha 3} + J_{\beta 3} - J_{\beta' 3}) + Q_{18}(J_{\alpha' 3} + J_{\gamma 3} - J_{\gamma' 3})$$

$$L_5 = 2(I_{\alpha 1} Q_{19} + I_{\alpha' 1} Q_{20}), \qquad L_6 = 2(I_{\alpha 3} Q_{21} + I_{\alpha' 3} Q_{22})$$

$$L_7 = 2(I_{\alpha 1} Q_{23} + I_{\alpha' 1} Q_{24}), \qquad L_8 = 2(I_{\alpha 3} Q_{25} + I_{\alpha' 3} Q_{26})$$

$$L_9 = (\ell_3 - \ell_4)\left[b_1' \frac{r_1}{h} \cot\beta (R_{\beta 1} Q_1 + R_{\beta' 1} Q_3) + b_3' \frac{r_1}{h} \cot\beta (R_{\beta 3} Q_{15} + R_{\beta' 3} Q_{27})\right.$$

$$+ b_5' \frac{r_1}{h} \cot\beta\,(R_{\beta 3} Q_{67} + R_{\beta 5} Q_{69}) + 2B_1'(Q_5 - Q_{83}) - 2B_2' Q_5 + 2B_3' Q_7$$

$$+ B_4'(3Q_7 + 2Q_9)] + m_1''[b_1' \frac{r_1}{h} \cot\beta\,(R_{\beta 1} Q_2 + R_{\beta 1}' Q_4) + b_3' \frac{r_1}{h} \cot\beta$$

$$\times (R_{\beta 3} Q_{23} + R_{\beta 3}' Q_{28}) + b_5' \frac{r_1}{h} \cot\beta\,(R_{\beta 5} Q_{68} + R_{\beta 5}' Q_{70}) + 2B_1'(Q_6$$

$$- Q_{84}) - 2B_2' Q_6 + 2B_3' Q_8 + B_4'(3Q_8 + 2Q_{10})]$$

$$L_{10} = (l_3 - l_4)[b_1' \frac{r_1}{h} \cot\beta\,(R_{\beta 1} Q_{30} + R_{\beta 1}' Q_{31}) + b_3' \frac{r_1}{h} \cot\beta\,(R_{\beta 3} Q_{32} + R_{\beta 3}' Q_{33})$$

$$+ b_5' \frac{r_1}{h} \cot\beta\,(R_{\beta 5} Q_{71} + R_{\beta 5}' Q_{72}) + 2B_1'(Q_{13} - Q_{85}) - 2B_2' Q_{13}$$

$$+ 2B_3' Q_{21} + B_4'(3Q_{21} + 2Q_{29})] + m_3''[b_1' \frac{r_1}{h} \cot\beta\,(R_{\beta 1} Q_{37}$$

$$+ R_{\beta 1}' Q_{38}) + b_3' \frac{r_1}{h} \cot\beta\,(R_{\beta 3} Q_{34} + R_{\beta 3}' Q_{35}) + b_5' \frac{r_1}{h} \cot\beta$$

$$\times (R_{\beta 5} Q_{73} + R_{\beta 5}' Q_{74}) + 2B_1'(Q_{14} - Q_{86}) - 2B_2' Q_{14} + 2B_3' Q_{22}$$

$$+ B_4'(3Q_{22} + 2Q_{36})]$$

$$L_{11} = (l_3 - l_4)[b_1' \frac{r_1}{h} \cot\beta\,(R_{\beta 1} Q_{42} + R_{\beta 1}' Q_{43}) + b_3' \frac{r_1}{h} \cot\beta\,(R_{\beta 3} Q_{75} + R_{\beta 3}' Q_{76})$$

$$+ b_5' \frac{r_1}{h} \cot\beta\,(R_{\beta 5} Q_{71} + R_{\beta 5}' Q_{72}) + 2B_1'(Q_{19} - Q_{84}) - 2B_2' Q_{13} + 2B_3' Q_{21}$$

$$+ B_4'(3Q_{21} + Q_{29})] + m_1''[b_1' \frac{r_1}{h} \cot\beta\,(R_{\beta 1} Q_{47} + R_{\beta 1}' Q_{48}) + b_3' \frac{r_1}{h} \cot\beta$$

$$\times (R_{\beta 3} Q_{39} + R'_{\beta 3} Q_{40}) + b'_5 \frac{\gamma_1}{\lambda} \cot\beta \, (R_{\beta 5} Q_{77} + R'_{\beta 5} Q_{78})$$

$$+ 2B'_1 (Q_{44} - Q_{87}) - 2B'_2 Q_{44} + 2B'_3 Q_{45} + B'_4 (3Q_{45} + 2Q_{46})] \}$$

$$L_{12} = (l_3 - l_4)[b'_1 \frac{\gamma_1}{\lambda} \cot\beta (R_{\beta 1} Q_{52} + R'_{\beta 1} Q_{53}) + b'_3 \frac{\gamma_1}{\lambda} \cot\beta (R_{\beta 3} Q_{41}$$

$$+ R'_{\beta 3} Q_{42}) + b'_5 \frac{\gamma_1}{\lambda} \cot\beta (R_{\beta 5} Q_{79} + R'_{\beta 5} Q_{80}) + 2B'_1 (Q_{49} - Q_{88})$$

$$- 2B'_2 Q_{49} + 3B'_3 Q_{50} + B'_4 (3Q_{50} + 2Q_{51})] + m''_3 [b'_1 \frac{\gamma_1}{\lambda} \cot\beta$$

$$\times (R_{\beta 1} Q_{57} + R'_{\beta 1} Q_{58}) + b'_3 \frac{\gamma_1}{\lambda} \cot\beta (R_{\beta 3} Q_{43} + R'_{\beta 3} Q_{63}) + b'_5 \frac{\gamma_1}{\lambda} \cot\beta$$

$$\times (R_{\beta 5} Q_{81} + R'_{\beta 5} Q_{82}) + 2B'_1 (Q_{54} - Q_{89}) - 2B'_2 Q_{54} + 2B'_3 Q_{55} + B'_4 (3Q_{55}$$

$$+ 2Q_{56})]$$

$$L_{13} = 0.5(\lambda^2 - \lambda - m''^2_1) \{ (l_2 - l_1) Q_{90} + 2[b'_1 \frac{\gamma_1}{\lambda} \cot\beta (I_{\beta 1} Q_1 + I'_{\beta 1} Q_3)$$

$$+ b'_3 \frac{\gamma_1}{\lambda} \cot\beta (I_{\beta 3} Q_{15} + I'_{\beta 3} Q_{27}) + b'_5 \frac{\gamma_1}{\lambda} \cot\beta (I_{\beta 5} Q_{67} + I'_{\beta 5} Q_{68})$$

$$- 2B'_1 (Q_5 - Q_{83}) + B'_2 Q_5 - 2B'_3 Q_7 + B'_4 (Q_7 + 2Q_9)] \} + 0.5(2\lambda - 1) m''_1$$

$$\times \{ (l_2 - l_1) Q_{91} + 2[b'_1 \frac{\gamma_1}{\lambda} \cot\beta (I_{\beta 1} Q_2 + I'_{\beta 1} Q_4) + b'_3 \frac{\gamma_1}{\lambda} \cot\beta$$

$$\times (I_{\beta 3} Q_{23} + I'_{\beta 3} Q_{28}) + b'_5 \frac{\gamma_1}{\lambda} \cot\beta (I_{\beta 5} Q_{68} + I'_{\beta 5} Q_{70}) - 2B'_1 (Q_6 - Q_{84})$$

$$+B_2'Q_6 - 2B_3'Q_8 + B_4'(Q_8 - 2Q_{10})]\Big\}$$

$$L_{14} = 0.5(\lambda^2 - \lambda - m_1''^2)\Big\{(l_2 - l_1)Q_{92} + 2\Big[b_1'\frac{r_1}{h}\cot\beta(I_{\beta 1}Q_{30} + I_{\beta 1}'Q_{31})$$

$$+ b_3'\frac{r_1}{h}\cot\beta(I_{\beta 3}Q_{32} + I_{\beta 3}'Q_{33}) + b_5'\frac{r_1}{h}\cot\beta(I_{\beta 5}Q_{71} + I_{\beta 5}'Q_{72})$$

$$- 2B_1'(Q_{13} - Q_{85}) + Q_2'Q_{13} - 2B_3'Q_{21} + B_4'(Q_{21} + Q_{29})]\Big\}$$

$$+ 0.5(2\lambda - 1)m_1''\Big\{(l_2 - l_1)Q_{93} + 2\Big[b_1'\frac{r_1}{h}\cot\beta(I_{\beta 1}Q_{37} + I_{\beta 1}'Q_{38})$$

$$+ b_3'\frac{r_1}{h}\cot\beta(I_{\beta 3}Q_{34} + I_{\beta 3}'Q_{35}) + b_5'\frac{r_1}{h}\cot\beta(I_{\beta 5}Q_{73} + I_{\beta 5}'Q_{74})$$

$$- 2B_1'(Q_{14} - Q_{86}) + B_2'Q_{14} - 2B_3'Q_{22} + B_4'(Q_{21} + 2Q_{29})]\Big\}$$

$$L_{15} = 0.5(\lambda^2 - \lambda - m_3''^2)\Big\{(l_2 - l_1)Q_{92} + 2\Big[b_1'\frac{r_1}{h}\cot\beta(I_{\beta 1}Q_{42} + I_{\beta 1}'Q_{43})$$

$$+ b_3'\frac{r_1}{h}\cot\beta(I_{\beta 3}Q_{75} + I_{\beta 3}'Q_{76}) + b_5'\frac{r_1}{h}\cot\beta(I_{\beta 5}Q_{71} + I_{\beta 5}'Q_{72})$$

$$- 2B_1'(Q_{13} - Q_{84}) + B_2'Q_{13} - 2B_3'Q_{21} + B_4'(Q_{21} + 2Q_{29})]\Big\}$$

$$+ 0.5\,m_3''(2\lambda - 1)\Big\{(l_2 - l_1)Q_{94} + 2\Big[b_1'\frac{r_1}{h}\cot\beta(I_{\beta 1}Q_{47} + I_{\beta 1}'Q_{48})$$

$$+ b_3'\frac{r_1}{h}\cot\beta(I_{\beta 3}Q_{39} + I_{\beta 3}'Q_{40}) + b_5'\frac{r_1}{h}\cot\beta(I_{\beta 5}Q_{77} + I_{\beta 5}'Q_{78})$$

$$- 2B_1'(Q_{44} - Q_{87}) + B_2'Q_{44} - 2B_3'Q_{45} - 2B_4'(Q_{45} + 2Q_{46})]\Big\}$$

$$L_{16} = 0.5(\lambda^2 - \lambda - m_3^{''2})\Big\{(l_2 - l_1)Q_{95} + 2\Big[b_1' \frac{r_1}{h}\cot\beta\,(I_{\beta 1}Q_{52} + I_{\beta 1}'Q_{53})$$

$$+ b_3' \frac{r_1}{h}\cot\beta\,(I_{\beta 3}Q_{41} + I_{\beta 3}'Q_{42}) + b_5' \frac{r_1}{h}\cot\beta\,(I_{\beta 5}Q_{79} + I_{\beta 5}'Q_{80})$$

$$- 2B_1'(Q_{49} - Q_{88}) - B_2'Q_{49} - 2B_3'Q_{50} + B_4'(Q_{21} + 2Q_{29})\Big]\Big\}$$

$$+ 0.5 m_3''(2\lambda - 1)\Big\{(l_2 - l_1)Q_{96} + 2\Big[b_1' \frac{r_1}{h}\cot\beta\,(I_{\beta 1}Q_{57} + I_{\beta 1}'Q_{58})$$

$$+ b_3' \frac{r_1}{h}\cot\beta\,(I_{\beta 3}Q_{43} + I_{\beta 3}'Q_{63}) + b_5' \frac{r_1}{h}\cot\beta\,(I_{\beta 5}'Q_{81} + I_{\beta 5}'Q_{82})$$

$$- 2B_1'(Q_{54} - Q_{89}) + B_2'Q_{54} - 2B_3'Q_{55} + B_4'(Q_{55} + 2Q_{56})\Big]\Big\}$$

$$i_{11} = DL_1 - \left(\frac{r_1^2}{h^2}\cot^2\beta\right)L_5 \quad , \qquad i_{13} = DL_2 - \left(\frac{r_1^2}{h^2}\cot^2\beta\right)L_6$$

$$i_{31} = DL_3 - \left(\frac{r_1^2}{h^2}\cot^2\beta\right)L_7 \quad , \qquad i_{33} = DL_4 - \left(\frac{r_1^2}{h^2}\cot^2\beta\right)L_8$$

$$k_{11} = -(L_9 + L_{13})\frac{r_1^2}{h^2} \quad , \qquad k_{13} = -(L_{10} + L_{14})\frac{r_1^2}{h^2}$$

$$k_{31} = -(L_{11} + L_{15})\frac{r_1^2}{h^2} \quad , \qquad k_{33} = -(L_{12} + L_{16})\frac{r_1^2}{h^2}$$

$$L_{17} = k_{11}k_{33} - k_{13}k_{31}, \qquad L_{18} = i_{11}i_{33} - i_{13}i_{31}$$

$$L_{19} = i_{11}k_{33} + i_{33}k_{11} - i_{13}k_{31} - i_{31}k_{13}$$

$$\alpha\tau_2 = 0.5\,L_{17}^{-1}\left[-L_{18} \pm (L_{18}^2 - 4L_{17}L_{19})^{0.5}\right]$$

APPENDIX C

COMPUTER PROGRAM FOR NONLINEAR ANALYSIS

CORRESPONDING SYMBOLS USED IN COMPUTER PROGRAM

ν = U, λ = A, μ = X, $\eta = \frac{1}{2}$ Q, x_o = C

$\frac{r_1}{h}$ = D, β = BETA, γ_1 = RATIO, ζ = PHI

g = SG, $\hat{a}_1'$ = A1H, $\hat{a}_3'$ = A2H, $\hat{A}_1'$ = A1P

$\hat{A}_2'$ = A2P, $\hat{A}_3'$ = A3P, $K_i(i=1,2,\ldots,49)$ = AI(I=1,2,...,49)

q_1 = D53 q_2 = D54 q_3 = D55, q_4 = D56

q_5 = D63 q_6 = D64 q_7 = D65, q_8 = D66

q_9 = D58 q_{10} = D59 q_{11} = D60 q_{12} = D61

$R_i(i=1,2,\ldots,10)$ = RI(I=1,2,...,10)

$S_i(i=1,2,\ldots,10)$ = SI(I=1,2,...,10)

V_1 = U6 V_2 = U7, V_3 = U8

σT_1 = AT01, AT02

```
C          MINIMUM TEMPERATURE IS OBTAINED BY APPLYING GALERKIN METHOD
C          TO THE GOVERNING DIFFERENTIAL EQUATIONS.
C          THE CONICAL SHELL IS SUBJECTED TO MERIDIONAL TEMPERATURE
C          GRADIENTS AND RESTRICTED COMPLETELY AT BOTH ENDS.
      U=0.3
      A=0.35
      EB=1./(12.*(1.-U*U))
    1 READ (5,10) BETA,D,X,Q,RATIO,SG,C,PHI
   10 FORMAT(8F10.5)
      EA=3.1416/C
      GA=3.*3.1416/C
      G=Q/SIN(BETA)
      E=TAN(BETA)
      F=X*3.1416/C
      FA=3.*F
      FP=3.*F
      FO=-1./(2.+SG)**2
      BT1=-(2.+3.*SG+SG*SG)/(SG+2.)**2
      BT2=-1./(SG+2.)
      P10=6.*A*A-12.*A+4.
      P20=4.*A**3-12.*A*A+8.*A
      P30=A**4-4.*A**3+4.*A*A
      AA=EA*EA+1.-A*A
      AB=EA**4+2.*(1.-3.*A*A)*EA*EA+(A*A-1.)**2
      AC=(AB*(EA*EA+A-A*A)-4.*(A-2.*A*A)*EA*EA*AA)/(AB*AB
     1+16.*EA*EA*A*A*AA*AA)
      AD=EA*((1.-2.*A)*AB+4.*A*(EA*EA+A-A*A)*AA)/(AB*AB+
     116.*EA*EA*A*A*AA*AA)
      AAP=GA*GA+1.-A*A
      ABP=GA**4+2.*(1.-3.*A*A)*GA*GA+(A*A-1.)**2
      ACP=(ABP*(GA*GA+A-A*A)-4.*(A-2.*A*A)*GA*GA*AAP)/(ABP*ABP
     1+16.*GA*GA*A*A*AAP*AAP)
      ADP=GA*((1.-2.*A)*ABP+4.*A*(GA*GA+A-A*A)*AAP)/(ABP*ABP+
     1 16.*GA*GA*A*A*AAP*AAP)
      AB1=(1.+A)**2*AD+2.*(1.+A)*EA*AC-EA*EA*AD
      AB2=(1.+A)*AD+EA*AC
      AB3=(1.+A)**2*AC-2.*(1.+A)*EA*AD-EA*EA*AC
      AB4=(1.+A)*AC-EA*AD
      AB1P=(1.+A)**2*ADP+2.*(1.+A)*GA*ACP-GA*GA*ADP
      AB2P=(1.+A)*ADP+GA*ACP
      AB3P=(1.+A)**2*ACP-2.*(1.+A)*GA*ADP-GA*GA*ACP
      AB4P=(1.+A)*ACP-GA*ADP
      TAU1=(2.-2.*U+3.*SG-U*SG+SG*SG)/(2.+SG)**2
      TAU2=(2.-2.*U+SG-3.*U*SG-U*SG*SG)/(2.+SG)**2/(1.+SG)
      ET1=EA/(EB*((1.+A+SG)**2+EA**2))*(1.+EXP((1.+A+SG)*C))
      ET2=GA/(EB*((1.+A+SG)**2+GA**2))*(1.+EXP((1.+A+SG)*C))
      AC1=(1.+EXP(A*C))/(A*A+EA*EA)*(EA*((1.+U)*AB4-U*
     1AB3)+A*(U*AB1-(1.+U)*AB2))
      AC2=(1.+U)*(1.-EXP(-C))/AC1
      AC3=2.*(1.-U)*(EXP(C)-1.)/AC1
      AC4=((1.-3.*U)*(EXP(C)-1.)+2.*(1.-U)*(1.+C*EXP(C)-EXP(C)))/AC1
```

```
 AC5=(1.-EXP(C))/AC1
 AC6=1./(AB1-(1.+U)*AB2)
 AC7=-(1.+U)*AC6
 AC8=2.*(1.-U)*AC6
 AC9=(3.-U)*AC6
 AC10=-AC7*EXP(-(1.+A)*C)
 AC11=-AC8*EXP((1.-A)*C)
 AC12=(-AC9-AC8*C)*EXP((1.-A)*C)
 AC13=AC6*EXP((1.-A)*C)
 W1=(1.+EXP(A*C))/(A*A+GA*GA)*(GA*((1.+U)*AB4P-U*
1AB3P)+A*(U*AB1P-(1.+U)*AB2P))/AC1
 W2=(AB1P-(1.+U)*AB2P)*AC6
 W3=W2
 AC5P=(1.+(1./(1.+SG)-TAU2)*(1.-EXP((1.+SG)*C))/(1.-EXP(C))
1*RATIO)*AC5
 AC6P=(1.+(1.-TAU1)*RATIO)*AC6
 AC13P=(1.+(1.-TAU1)*EXP(SG*C)*RATIO)*AC13
 AD1=(AC5P+AC6P)/(AC2-AC7)
 AD2=(AC3-AC8)/(AC2-AC7)
 AD3=(AC4-AC9)/(AC2-AC7)
 AD4=(AC5P-AC13P)/(AC2-AC10)
 AD5=(AC3-AC11)/(AC2-AC10)
 AD6=(AC4-AC12)/(AC2-AC10)
 AD7=(AD1-AD4)/(AD2-AD5)
 AD8=(AD6-AD3)/(AD2-AD5)
 AD9=AD1-AD2*AD7
 AD10=-(AD2*AD8+AD3)
 AD11=-1./(AC2*AD10+AC3*AD8+AC4)
 AD12=(AC2*AD9+AC3*AD7-AC5P)*AD11
 AD13=AD10*AD11
 AD14=AD9+AD10*AD12
 AD15=AD8*AD11
 ADD6=AD7+AD8*AD12
 AD16=(W1-W2)/(AC2-AC7)
 AD17=(W1-W3)/(AC2-AC10)
 AD18=(AD17-AD16)/(AD2-AD5)
 AD19=-(AD2*AD18+AD16)
 AD20=(W1+AC2*AD19+AC3*AD18)*AD11
 AD21=AD19+AD10*AD20
 AD22=AD18+AD8*AD20
 AE1=(EA**4-P10*EA*EA+P30)*((EXP(2.*(A-1.)*C)-1.)/4./(A-1.)
1-(A-1.)*(EXP((2.*A-2.)*C)-1.)/(4.*(A-1.)**2+4.*EA*EA))
 AE2=(4.*(1.-A)*EA**4+P20*EA*EA)*(1.-EXP((2.*A-2.)*C))
1/(4.*(A-1.)**2+4.*EA*EA)
 AE3=(AB4-AB3)/E8*((EXP(2.*A*C)-1.)/4./A-(EXP(2.*A*C)-1.)
1/(4.*A*A+4.*EA*EA)*A)
 AE4=(AB2-AB1)/E8*EA*(1.-EXP(2.*A*C))/(4.*A*A+4.*EA*EA)
 AE1P=(GA**4-P10*GA*GA+P30)*(A-1.)*(-EXP(2.*(A-1.)*C)+1.)*(1./
1(4.*(A-1.)**2+(EA+GA)**2)-1./(4.*(A-1.)**2+(EA-GA)**2))
 AE2P=(4.*(1.-A)*GA**4+P20*GA*GA)*(-EXP(2.*(A-1.)*C)+1.)/2.*((EA+GA
1)/(4.*(A-1.)**2+(EA+GA)**2)-(GA-EA)/(4.*(A-1.)**2+(GA-EA)**2))
```

```
 AE3P=(AB4P-AB3P)/EB*A*(-EXP(2.*A*C)+1.)*(1./(4.*A*A+(EA+GA)**2)-
11./(4.*A*A+(EA-GA)**2))
 AE4P=(AB2P-AB1P)/EB*(-EXP(2.*A*C)+1.)/2.*((EA+GA)/(4.*A*A+
1(EA+GA)**2)-(GA-EA)/(4.*A*A+(GA-EA)**2))
 AE5=EA/EB*(EXP((A-1.)*C)+1.)/((A-1.)**2+EA*EA)
 AE6=-EA/EB*(EXP((A+1.)*C)+1.)/((A+1.)**2+EA*EA)
 AE7=-2./EB*(A+1.)**2/((A+1.)**2+EA*EA)*(-EA/(A+1.)
1*(EXP((A+1.)*C)+1.)/((A+1.)**2+EA*EA)+EA/(A+1.)*(C/(A+1.)
2-1./(A+1.)**2)*EXP((A+1.)*C)-EA/(A+1.)**3+EA**3
3 /(A+1.)**3*(1.+EXP((1.+A)*C))/((1.+A)**2+EA*EA))
 AE8=AE5*AD14+2.*AE6*ADD6+3.*AE6*AD12+AE7*AD12
 AE9=AE1+AE2
 AE10=AE7*AD11+AE3+AE4+AE5*AD13+2.*AE6*AD15+3.*AE6*AD11
 AE11=(EA**4-P10*EA*EA+P30)*(A-1.)*(-EXP(2.*(A-1.)*C)+1.)*(
11./(4.*(A-1.)**2+(EA+GA)**2)-1./(4.*(A-1.)**2+(EA-GA)**2))
 AE12=(4.*(1.-A)*EA**4+P20*EA**2)*(-EXP(2.*(A-1.)*C)+1.)/2.
1*((EA+GA)/(4.*(A-1.)**2+(EA+GA)**2)+(GA-EA)/(4.*(A-1.)**2+(GA-EA)
2**2))
 AE11P=(GA**4-P10*GA**2+P30)*(EXP(2.*(A-1.)*C)-1.)*(1./(4.*(A-1.
1))-(A-1.)/(4.*(A-1.)**2+4.*GA*GA))
 AE12P=(4.*(1.-A)*GA**4+P20*GA**2)*(1.-EXP(2.*(A-1.)*C))/(4.*(A-1.)
1**2+4.*GA*GA)
 AE13=(AB4-AB3)/EB*A*(-EXP(2.*A*C)+1.)*(1./(4.*A*A+(EA+GA)**2)-
11./(4.*A*A+(EA-GA)**2))
 AE14=(AB2-AB1)/EB*(-EXP(2.*A*C)+1.)/2.*((EA+GA)/(4.*A*A+(EA+GA)**2
1)+(GA-EA)/(4.*A*A+(GA-EA)**2))
 AE13P=(AB4P-AB3P)/EB*(EXP(2.*A*C)-1.)*(1./4./A-A/(4.*A*A+4.*GA
1*GA))
 AE14P=(AB2P-AB1P)/EB*GA*(1.-EXP(2.*A*C))/(4.*A*A+4.*GA*GA)
 AE15=GA/EB*(EXP((A-1.)*C)+1.)/((A-1.)**2+GA*GA)
 AE16=-GA/EB*(EXP((A+1.)*C)+1.)/((A+1.)**2+GA*GA)
 AE17=-2./EB*(A+1.)**2/((A+1.)**2+GA*GA)*(-GA/(A+1.)*(EXP((A+1.)*C)
1+1.)/((A+1.)**2+GA*GA)-GA/(A+1.)*(C/(A+1.)-1./(A+1.)**2)*EXP((
2A+1.)*C)-GA/(A+1.)**3+GA**3/(A+1.)**3*(1.+EXP((1.+A)*C))/((1.+A)
3**2+GA**2))
 AE18=AE1P+AE2P
 AE19=AE3P+AE4P+AD21*AE5+2.*AE6*AD22+3.*AE6*AD20+AE7*AD20
 AE20=AE11+AE12
 AE21=AE13+AE14+AE15*AD13+2.*AE16*AD15+3.*AE16*AD11+AE17*AD11
 AE22=AE11P+AE12P
 AE23=AE13P+AE14P+AE15*AD21+2.*AE16*AD22+3.*AE16*AD20+AE17*AD20
 AE24=AE15*AD14+2.*AE16*AD16+3.*AE16*AD12+AE17*AD12
 AE25=AE9+D*D/E**2*AE10
 AE26=-(AE18+D*D/E**2*AE19)/AE25
 AE27P=-(AE8*D/E-BT1*ET1*RATIO*D/E)/AE25
 AE27=AE27P
 AE28=AE20+D*D/E**2*AE21
 AE29=-(AE22+D*D/E**2*AE23)/AE28
 AE30P=-(AE24*D/E-BT1*ET2*RATIO*D/E)/AE28
 AE30=AE30P
 AE31=(AE27-AE30)/(AE29-AE26)
```

```
AE32=AE26*AE31+AE27
A1H=AE32
A2H=AE31
A1P=AD14+D/E*(AD21*AE31+AD13*AE32)
A2P=ADD6+D/E*(AE31*AD22+AE32*AD15)
A3P=AD12+D/E*(AD20*AE31+AD11*AE32)
B1=-F*F
B2=2.*F*A-F
B3=-FA*FA
B4=FA*(2.*A-1.)
B5=-EA*EA
B6=2.*A*EA-EA
B7=-GA*GA
B8=GA*(2.*A-1.)
B9=-G*G
B10=-A1H*B5*B9-A1H*B1*A
B11=-A1H*B6*B9-A1H*B1*EA
B12=-B7*B9*A2H-A2H*B1*A
B13=-A2H*B8*B9-A2H*B1*GA
B14=-A1H*B5*F-A1H*B2*A
B15=-A1H*B6*F-A1H*B2*EA
B16=-A2H*B7*F-A2H*B2*A
B17=-A2H*B8*F-A2H*B2*GA
B18=-A1H*B5*B9-A1H*B3*A
B19=-A1H*B6*B9-A1H*B3*EA
B20=-A2H*B7*B9-A2H*B3*A
B21=-A2H*B8*B9-A2H*B3*GA
B22=-A1H*B5*FA-A1H*B4*A
B23=-A1H*B6*FA-A1H*B4*EA
B24=-A2H*B7*FA-A2H*B4*A
B25=-A2H*B8*FA-A2H*B4*GA
B26=-A1H*A1H*B5*A
B27=-A1H*A1H*B6*A-A1H*A1H*B5*EA
B28=-A1H*A2H*B7*A-A1H*A2H*B5*A
B29=-A1H*A2H*B8*A-A1H*A2H*B5*GA
B30=-A1H**2*B6*EA
B31=-A1H*A2H*(B7*EA+B6*A)
B32=-A1H*A2H*(B8*EA+B6*GA)
B33=-A2H**2*B7*A
B34=-A2H**2*(B8*A+B7*GA)
B35=-A2H**2*B8*GA
B36=-B1*B9
B37=-B2*B9-B1*F
B38=-B3*B9-B1*B9
B39=-B4*B9-B1*FA
B40=-B2*F
B41=-B3*F-B2*B9
B42=-B4*F-B2*FA
B43=-B3*B9
B44=-B4*B9
B45=-B3*FA
```

```
B46=-B4*FA
B47=G*G*(A*A-2.*A+1.)
B48=G*G*(A*A-2.*A+1.)
B49=G*G*(2.*A*F-2.*F)
B50=G*G*(2.*A*A-4.*A+2.)
B51=G*G*(2.*A*FA-2.*FA)
B52=G*G*(2.*F*A-2.*F)
B53=2.*G*G*F*FA
B54=G*G*(2.*A*FA-2.*FA)
B55=0.5*(B10+B15)
B56=.5*(B11-B14)
B57=.5*(-B10+B15)
B58=.5*(B11+B14)
B59=.5*(B12+B17)
B60=.5*(B13-B16)
B61=.5*(-B12+B17)
B62=.5*(B13+B16)
B63=.5*(B18+B23)
B64=.5*(B19-B22)
B65=.5*(-B18+B23)
B66=.5*(B19+B22)
B67=.5*(B20-B25)
B68=.5*(B21-B24)
B69=.5*(-B20+B25)
B70=.5*(B21+B24)
B71=.5*(B26+B30+B33+B35)
B72=.5*(-B26+B30)
B73=.5*B27
B74=.5*(B28+B32)
B75=.5*(-B29+B31)
B76=.5*(-B28+B32)
B77=.5*(B29+B31)
B78=.5*(-B33+B35)
B79=.5*B34
B80=.25*(B36+B40+B47+F*F)
B81=.25*(-B36+B40-B47+F*F)
B82=.25*(B37+B49)
B83=.25*(B36+B40-B47-F*F)
B84=.25*(-B36+B40+B47-F*F)
B85=.25*(B37-B49)
B86=.25*(B38+B42+B50+B53)
B87=.25*(-B39+B41-B51+B52)
B88=.25*(-B38+B42-B50+B53)
B89=.25*(B39+B41+B51+B52)
B90=.25*(B38+B42-B50-B53)
B91=.25*(-B39+B41+B51-B52)
B92=.25*(-B38+B42+B50-B53)
B93=.25*(B39+B41-B51-B52)
B94=.25*(B43+B46+B48+FA*FA)
B95=.25*(-B43+B46-B48+FA*FA)
B96=.25*(B44+B45+B54)
```

```
B97=.25*(B43+B46-B48-FA*FA)
B98=.25*(-B43+B46+B48-FA*FA)
B99=.25*(B44+B45-B54)
B100=-EA**2
B101=EA*(2.*A-1.)
B102=-GA**2
B103=GA*(2.*A-1.)
B104=-F*F
B105=F*(2.*A-1.)
B106=-FA*FA
B107=FA*(2.*A-1.)
C1=(F*F+G*G)**2
C2=-4.*(1.+A)*F**3+4.*F**3+4.*F*G*G-4.*(1.+A)*G*G*F
C3=(FA*FA+G*G)**2
C4=-4.*(1.+A)*FA**3+4.*FA**3+4.*FA*G*G-4.*FA*(1.+A)*G*G
C5=((F-EA)**2+G*G)**2
C6=-8.*A*(F-EA)**3+4.*(F-EA)**3+4.*(F-EA)*G*G-8.*A*G*(F-EA)*G
C7=((F+EA)**2+G*G)**2
C8=-8.*A*(F+EA)**3+4.*(F+EA)*G*G-8.*A*G*G*(F+EA)+4.*(F+EA)**3
C9=((F-GA)**2+G*G)**2
C10=-8.*A*(F-GA)**3+4.*(F-GA)**3+4.*(F-GA)*G*G-8.*A*G*G*(F-GA)
C11=((F+GA)**2+G*G)**2
C12=-8.*A*(F+GA)**3+4.*(F+GA)**3+4.*(F+GA)*G*G-8.*A*G*G*(F+GA)
C13=((FA-EA)**2+G*G)**2
C14=-8.*A*(FA-EA)**3+4.*(FA-EA)**3+4.*(FA-EA)*G*G-8.*A*G*G*(FA-EA)
C15=((FA+EA)**2+G*G)**2
C16=-8.*A*(FA+EA)**3+4.*(FA+EA)**3+4.*(FA+EA)*G*G-8.*A*G*G*(FA+EA)
C17=((FA-GA)**2+G*G)**2
C18=-8.*A*(FA-GA)**3+4.*(FA-GA)**3+4.*(FA-GA)*G*G-8.*A*G*G*(FA-GA)
C19=((FA+GA)**2+G*G)**2
C20=-8.*A*(FA+GA)**3+4.*(FA+GA)**3+4.*(FA+GA)*G*G-8.*A*G*G*(GA+FA)
C21=16.*A**4-32.*A**3+16.*A*A
C22=16.*EA**4
C23=-64.*A*EA**3+32.*EA**3
C24=(GA-EA)**4
C25=-8.*A*(GA-EA)**3+4.*(GA-EA)**3
C26=(GA+EA)**4
C27=-8.*A*(GA+EA)**3+4.*(GA+EA)**3
C28=16.*GA**4
C29=-64.*A*GA**3+32.*GA**3
C30=16.*A**4-32.*A**3+16.*A**2
C31=16.*F **4
C32=-64.*A*F**3+32.*F**3
C33=16.*G**4
C34=(4.*F**2+4.*G*G)**2
C35=-64.*A*F**3+32.*F**3+32.*F*G*G-64.*A*F*G*G
C36=(FA-F)**4
C37=-8.*A*(FA-F)**3+4.*(FA-F)**3
C38=(FA+F)**4
C39=-8.*A*(F+FA)**3+4.*(F+FA)**3
C40=((FA-F)**2+4.*G*G)**2
```

```
C41=-8.*A*(FA-F)**3+4.*(FA-F)**3+16.*(FA-F)*G*G-32.*A*G*G*(FA-F)
C42=((F+FA)**2+4.*G*G)**2
C43=-8.*A*(F+FA)**3+4.*(F+FA)**3+16.*(F+FA)*G*G-32.*A*G*G*(F+FA)
C44=16.*A**4-32.*A**3+16.*A**2
C45=16.*FA**3
C46=-64.*A*FA**3+32.*FA**3
C47=16.*G**4
C48=(4.*FA**2+4.*G*G)**2
C49=-64.*A*FA**3+32.*FA**3+32.*FA*G*G-64.*A*G*G*FA
A1=D/E*(-B2*C1+C2*B1)/(C1**2+C2**2)
A2=D/E*(-C1*B1-B2*C2)/(C1**2+C2**2)
A3=D/E*(-B4*C3+B3*C4)/(C3**2+C4**2)
A4=D/E*(-C3*B3-B4*C4)/(C3**2+C4**2)
A5=(B55*C5-B56*C6)/(C5**2+C6**2)
A6=(B56*C5+B55*C6)/(C5**2+C6**2)
A7=(C7*B57-B58*C8)/(C7**2+C8**2)
A8=(C7*B58+B57*C8)/(C7**2+C8**2)
A9=(B59*C9-B60*C10)/(C9**2+C10**2)
A10=(C9*B60+B59*C10)/(C9**2+C10**2)
A11=(B61*C11-B62*C12)/(C11**2+C12**2)
A12=(B62*C11+B61*C12)/(C11**2+C12**2)
A13=(C13*B63-B64*C14)/(C13**2+C14**2)
A14=(C13*B64+C14*B63)/(C13**2+C14**2)
A15=(B65*C15-C16*B66)/(C15**2+C16**2)
A16=(C15*B66+B65*C16)/(C15**2+C16**2)
A17=(C17*B67-C18*B68)/(C17**2+C18**2)
A18=(C17*B68+B67*C18)/(C17**2+C18**2)
A19=(C19*B69-C20*B70)/(C19**2+C20**2)
A20=(C19*B70+B69*C20)/(C19**2+C20**2)
A21=B71/C21
A22=(B72*C22-C23*B73)/(C22**2+C23**2)
A23=(C22*B73+C23*B72)/(C22**2+C23**2)
A24=(C24*B74-C25*B75)/(C24**2+C25**2)
A25=(C24*B75+C25*B74)/(C24**2+C25**2)
A26=(C26*B76-C27*B77)/(C26**2+C27**2)
A27=(C26*B77+C27*B76)/(C26**2+C27**2)
A28=(C28*B78-C29*B79)/(C28**2+C29**2)
A29=(C28*B79+C29*B78)/(C28**2+C29**2)
A30=B80/C30
A31=(C31*B81-C32*B82)/(C31**2+C32**2)
A32=(C31*B82+C32*B81)/(C31**2+C32**2)
A33=B83/C33
A34=(C34*B84-C35*B85)/(C34**2+C35**2)
A35=(C34*B85+C35*B84)/(C34**2+C35**2)
A36=(C36*B86-C37*B87)/(C36**2+C37**2)
A37=(C36*B87+C37*B86)/(C36**2+C37**2)
A38=(C38*B88-C39*B89)/(C38**2+C39**2)
A39=(C38*B89+C39*B88)/(C38**2+C39**2)
A40=(C40*B90-C41*B91)/(C40**2+C41**2)
A41=(C40*B91+C41*B90)/(C40**2+C41**2)
A42=(B92*C42-C43*B93)/(C42**2+C43**2)
```

```
 A43=(C42*B93+C43*B92)/(C42**2+C43**2)
 A44=B94/C44
 A45=(C45*B95-B96*C46)/(C45**2+C46**2)
 A46=(C45*B96+C46*B95)/(C45**2+C46**2)
 A47=B97/C47
 A48=(C48*B98-C49*B99)/(C48**2+C49**2)
 A49=(C48*B99+C49*B98)/(C48**2+C49**2)
 C50=(4.*A*A-(1.+U)*2.*A)*A21+ 4.*A*A*A22+8.*A23*A*EA-4.*A22*EA**
12-(1.+U)*(2.*A23*EA+2.*A22*A)+4.*A*A*A24+4.*A25*(GA-EA)-A24*
2(GA-EA)**2-(1.+U)*(A25*(GA-EA)+2.*A24*A)+4.*A*A*A26+4.*A27*(GA
3+EA)-A26*(GA+EA)**2-(1.+U)*(A27*(GA+EA)+2.*A26*A)+4.*A*A*A28
4+8.*A29*GA-4.*A28*GA*GA-(1.+U)*(2.*A29*GA+2.*A28*A)
 C51=A30*(4.*A*A-(1.-U)*2.*A)+4.*A*A*A31+8.*A32*F*A-4.*A31*F*F-
1(1.+U)*(2.*A32*F+2.*A31*A)
 C52=4.*A*A*A36+2.*A37*(FA-F)**2-A36*(FA-F)**2-(1.+U)*((FA-F)
1*A37+2.*A36*A)+4.*A*A*A38+2.*A39*(FA+F)**2-A38*(FA+F)**2-(1.+U)
2*(A39*(FA+F)+2.*A38*A)
 C53=A44*(4.*A*A-(1.+U)*2.*A)+4.*A*A*A45+8.*A46*FA*A-4.*A45*FA
1*FA-(1.+U)*(2.*A46*FA+2.*A45*A)
 C54=0.
 C55=C50*EXP(2.*(A-1.)*C)
 C56=C51*EXP(2.*(A-1.)*C)
 C57=C52*EXP(2.*(A-1.)*C)
 C58=C53*EXP(2.*(A-1.)*C)
 C59=EXP(SG*C)*C54
 C60=A21*((1.+U)*2.*A-4.*U*A*A)
 C61=(1.+U)*(2.*A23*A-2.*A22*EA)-U*(4.*A23*A*A-8.*A22*A*EA-4.
1*A23*EA*EA)
 C62=(1.+U)*(2.*A23*EA+2.*A22*A)-U*(4.*A22*A*A+8.*A23*A*EA-4.
1*A22*EA*EA)
 C63=(1.+U)*(2.*A25*A-A24*(GA-EA))-U*(4.*A25*A*A-4.*A24*A*(GA-EA)
1-A25*(GA-EA)**2)
 C64=(1.+U)*(A25*(GA-EA)+2.*A24*A)-U*(4.*A24*A*A+4.*A25*A*(GA
1-EA)-A24*(GA-EA)**2)
 C65=(1.+U)*(2.*A27*A-A26*(GA+EA))-U*(4.*A27*A*A-4.*A26*A*(GA+
1EA)-A27*(EA+GA)**2)
 C66=(1.+U)*(A27*(EA+GA)+2.*A26*A)-U*(4.*A26*A*A+4.*A27*A*(EA+GA)
1-A26*(EA+GA)**2)
 C67=(1.+U)*(2.*A29*A-2.*A28*GA)-U*(4.*A29*A*A-8.*A28*A*GA-4.*A29
1*GA*GA)
 C68=(1.+U)*(2.*A29*GA+2.*A28*A)-U*(4.*A28*A*A+8.*A29*A*GA-4.
1*A28*GA*GA)
 C70=A30*((1.+U)*2.*A-4.*U*A*A)
 C71=(1.+U)*(2.*A32*A-2.*A31*F)-U*(4.*A32*A*A-8.*A31*A*F-4.*A32*F*
1F)
 C72=(1.+U)*(2.*A32*F+2.*A31*A)-U*(4.*A31*A*A+8.*A32*A*F-4.*A31*
1F*F)
 C73=(1.+U)*(2.*A37*A-A36*(FA-F))-U*(4.*A37*A*A-4.*A36*A*(FA-F)-
1A37*(FA-F)**2)
 C74=(1.+U)*(A37*(FA-F)+2.*A36*A)-U*(4.*A36*A*A+4.*A37*A*(FA-F)
1-A36*(FA-F)**2)
```

```
 C75=(1.+U)*(2.*A39*A-A38*(FA+F))-U*(4.*A39*A*A-4.*A38*A*(FA+F)
1-A39*(FA+F)**2)
 C76=(1.+U)*(A39*(FA+F)+2.*A38*A)-U*(4.*A38*A*A+4.*A37*A*(FA+F)
1-A38*(FA+F)**2)
 C77=A44*((1.+U)*2.*A-4.*A*A*U)
 C78=(1.+U)*(2.*A46*A-2.*A45*FA)-U*(4.*A46*A*A-8.*A45*A*FA-4.*A46
1*FA*FA)
 C79=(1.+U)*(2.*A46*FA+2.*A45*A)-U*(4.*A45*A*A+8.*A46*A*FA
1-4.*A45*FA*FA)
 DA=EXP((2.*A-1.)*C)-1.
 D1=DA/(2.*A-1.)
 D2=-DA/2./EA
 D3=DA/4./EA**2*(2.*A-1.)
 D4=-DA/(GA-EA)
 D5=DA*(2.*A-1.)/(GA-EA)**2
 D6=-DA/(GA+EA)
 D7=DA*(2.*A-1.)/(GA+EA)**2
 D8=-DA/2./GA
 D9=DA/4./GA**2*(2.*A-1.)
 D11=-DA/2./F
 D12=DA/4./F**2 *(2.*A-1.)
 D13=-DA/(FA-F)
 D14=DA/(FA-F)**2*(2.*A-1.)
 D15=-DA/(FA+F)
 D16=DA*(2.*A-1.)/(FA+F)**2
 D17=-DA/2./FA
 D18=DA/4./FA**2*(2.*A-1.)
 D19=(1.+U)*(1.-EXP(-C))
 D20=2.*(1.+U)*(EXP(C)-1.)
 D21=(1.-3.*U)*(EXP(C)-1.)+2.*(1.-U)*(1.+C*EXP(C)-EXP(C))
 D22=0.
 D23=A1H**2*A*A/2./(2.*A-1.)*DA+A1H**2*EA*EA/2./(2.*A-1.)*DA+A2H
1**2*A*A/2./(2.*A-1.)*DA+A2H**2*GA**2/2./(2.*A-1.)*DA+A1H**2*A/2.*
2(-DA)+A*A*(1.-2.*A)*(-DA)*(1./(GA-EA)**2-1./(GA+EA)**2)*A1H*A2H
3+A1H*A2H*A*GA*(-DA)*(1./(EA+GA)-1./(GA-EA))+A1H*A2H*A*EA*(-DA)
4*(1./(EA+GA)+1./(GA-EA))+A1H*A2H*(1.-2.*A)*(-DA)*EA*GA*(1.
5/(EA+GA)**2+1./(EA-GA)**2)+A/2.*A2H**2*(-DA)
 D24=.5*(A*A/2./(2.*A-1.)*DA+F*F*DA/2./(2.*A-1.)+A/2.*(-DA))
 D25=.5*(A*A*(2.*A-1.)*DA*(1./(FA-F)**2-1./(F+FA)**2)+A*FA*(-DA)
1*(1./(F+FA)-1./(FA-F))+A*F*(-DA)*(1./(F+FA)+1./(FA-F))
2+(2.*A-1.)*F*FA*DA*(1./(F+FA)**2+1./(FA-F)**2))
 D26=.5*(A*A/2./(2.*A-1.)*DA+FA**2*DA/2./(2.*A-1.)+A/2.*(-DA))
 D27=D24/2.-(D1*C70+C71*D11+C72*D12)
 D28=D25/2.-(C73*D13+C74*D14+C75*D15+C76*D16)
 D29=D26/2.-(C77*D1+C78*D17+C79*D18)
 D30=-(C60*D1+C61*D2+C62*D3+C63*D4+C64*D5+C65*D6+C66*D7+C67*D8
1+C68*D9)+D23/2.
 D31=-D22
 D32=1.+U
 D33=2.*(U-1.)
 D34=U-3.
```

```
D35=(1.+U)*EXP(-2.*C)
D36=D33
D37=-(3.-U+2.*C-2.*U*C)
D38=D35-D32
D39=D37-D34
D40=C56-C51
D41=C57-C52
D42=C58-C53
D43=C55-C50
D44=C59-C54
D45=D19*D33-D32*D20
D46=D21*D33-D34*D20
D47=D27*D33-C51*D20
D48=D33*D28-C52*D20
D49=D29*D33-D20*C53
D50=D33*D30-C50*D20
D51=D31*D33-C54*D20
D52=D38*D46-D39*D45
D53=(D40*D46-D47*D39)/D52
D54=(D41*D46-D48*D39)/D52
D55=(D42*D46-D49*D39)/D52
D56=(D43*D46-D39*D50)/D52
D57=(D44*D46-D39*D51)/D52
D58=(D47-D53*D45)/D46
D59=(D48-D54*D45)/D46
D60=(D49-D55*D45)/D46
D61=(D50-D56*D45)/D46
D62=(D51-D57*D45)/D46
D63=(D27-D53*D19-D58*D21)/D20
D64=(D28-D54*D19-D59*D21)/D20
D65=(D29-D55*D19-D60*D21)/D20
D66=(D30-D56*D19-D61*D21)/D20
D67=(D31-D57*D19-D62*D21)/D20
E1=(F*F+G*G)**2
E2=-4.*F**3*A+4.*F**3+4.*F*G*G-4.*F*G*G*A
E3=(FA**2+G*G)**2
E4=-4.*FA**3*A+4.*FA**3+4.*FA*G*G-4.*FA*G*G*A
E5=-2.*A1*(1.+A)*F-A2*F*F+A1*F
E6=2.*A2*(1.+A)*F-A1*F*F-A2*F
E7=-2.*A3*(1.+A)*FA-A4*FA**2+A3*FA
E8=2.*A4*(1.+A)*FA-A3*FA**2-A4*FA
E9=-4.*A5*A*(F-EA)-A6*(F-EA)**2+A5*(F-EA)
E10=4.*A6*A*(F-EA)-A5*(F-EA)**2-A6*(F-EA)
E11=-4.*A7*A*(F+EA)-A8*(F+EA)**2+A7*(F+EA)
E12=4.*A8*A*(F+EA)-A7*(F+EA)**2-A8*(F+EA)
E13=-4.*A9*A*(F-GA)-A10*(F-GA)**2+A9*(F-GA)
E14=4.*A10*A*(F-GA)-A9*(F-GA)**2-A10*(F-GA)
E15=-4.*A11*A*(F+GA)-A12*(F+GA)**2+A11*(F+GA)
E16=4.*A12*A*(F+GA)-A11*(F+GA)**2-A12*(F+GA)
E17=-4.*A13*A*(FA-EA)-A14*(FA-EA)**2+A13*(FA-EA)
E18=4.*A14*A*(FA-EA)-A13*(FA-EA)**2-A14*(FA-EA)
```

```
E19=-4.*A15*A*(FA+EA)-A16*(FA+EA)**2+A15*(FA+EA)
E20=4.*A16*A*(FA+EA)-A15*(FA+EA)**2-A16*(FA+EA)
E21=-4.*A17*A*(FA-GA)-A18*(FA-GA)**2+A17*(FA-GA)
E22=4.*A18*A*(FA-GA)-A17*(FA-GA)**2-A18*(FA-GA)
E23=-4.*A*A19*(FA+GA)-A20*(FA+GA)**2+A19*(FA+GA)
E24=4.*A*A20*(FA+GA)-A19*(FA+GA)**2-A20*(FA+GA)
E25=A21*(4.*A*A-2.*A)
E26=-8.*A22*A*EA-4.*A23*EA**2+2.*A22*EA
E27=8.*A23*A*EA-4.*A22*EA**2-2.*A23*EA
E28=-4.*A24*A*(GA-EA)-A25*(GA-EA)**2+A24*(GA-EA)
E29=4.*A25*A*(GA-EA)-A24*(GA-EA)**2-A25*(GA-EA)
E30=-4.*A26*A*(GA+EA)-A27*(GA+EA)**2+A26*(GA+EA)
E31=4.*A27*A*(GA+EA)-A26*(GA+EA)**2-A27*(GA+EA)
E32=-8.*A28*A*GA-4.*A29*GA*GA+2.*A28*GA
E33=8.*A29*A*GA-4.*A28*GA**2-2.*A29*GA
E34=A30*(4.*A*A-2.*A)
E35=-8.*A31*A*F-4.*A32*F*F+2.*A31*F
E36=8.*A32*A*F-4.*A31*F*F-2.*A32*F
E37=-4.*A36*A*(FA-F)-A37*(FA-F)**2+A36*(FA-F)
E38=4.*A37*A*(FA-F)-A36*(FA-F)**2-A37*(FA-F)
E39=-4.*A38*A*(FA+F)-A39*(FA+F)**2+A38*(FA+F)
E40=4.*A39*A*(FA+F)-A38*(FA+F)**2-A39*(FA+F)
E41=A44*(4.*A*A-2.*A)
E42=-8.*A45*A*FA-4.*A46*FA**2+2.*A45*FA
E43=8.*A46*A*FA-4.*A45*FA**2-2.*A46*FA
E44=A33*(4.*A*A-2.*A)
E45=-8.*A35*A*F-4.*A36*F*F+2.*A35*F
E46=8.*A36*A*F-4.*A35*F*F-2.*A36*F
E47=-4.*A40*A*(FA-F)-A41*(FA-F)**2+A40*(FA-F)
E48=4.*A41*A*(FA-F)-A40*(FA-F)**2-A41*(FA-F)
E49=-4.*A42*A*(FA+F)-A43*(FA+F)**2+A42*(FA+F)
E50=4.*A43*A*(FA+F)-A42*(FA+F)**2-A43*(FA+F)
E50P=A47*(4.*A*A-2.*A)
E51=-8.*A48*A*FA-4.*A49*FA**2+2.*A48*FA
E52=8.*A49*A*FA-4.*A48*FA*FA-2.*A49*FA
E53=((2.+SG)**2-(2.+SG))*F0*RATIO
E54=(-2.*AD*(1.+A)*EA-AC*EA*EA+AD*EA)*A1H
E55=(2.*AC*(1.+A)*EA-AD*EA*EA-AC*EA)*A1H
E56=(-2.*ADP*(1.+A)*GA-ACP*GA*GA+ADP*GA)*A2H
E57=(2.*ACP*(1.+A)*GA-ADP*GA*GA-ACP*GA)*A2H
E58=2.*G*(1.-2.*A)*A33
E59=2.*G*(2.*A34*F-2.*A*A35+A35)
E60=2.*G*(-2.*A*A34-2.*F*A35+A34)
E61=(A40*(FA-F)-2.*A*A41+A41)*2.*G
E62=2.*G*(-2.*A*A40-(FA-F)*A41+A40)
E63=2.*G*(A42*(FA+F)-2.*A*A43+A43)
E64=2.*G*(-2.*A*A42-(FA+F)*A43+A42)
E65=2.*G*(1.-2.*A)*A47
E66=(2.*FA*A48-2.*A*A49+A49)*2.*G
E67=(-2.*A*A48-2.*FA*A49+A48)*2.*G
E68=(1.+A)*A2-A1*F-A2*G*G
```

```
E69=(1.+A)*A1+F*A2-A1*G*G
E70=(1.+A)*A4-A3*FA-A4*G*G
E71=(1.+A)*A3+A4*FA-A3*G*G
E72=2.*A*A6-A5*(F-EA)-A6*G*G
E73=2.*A*A5+(F-EA)*A6-A5*G*G
E74=2.*A*A8-A7*(F+EA)-A8*G*G
E75=2.*A7*A+(F+EA)*A8-A7*G*G
E76=2.*A*A10-A9*(F-GA)-A10*G*G
E77=2.*A9*A+(F-GA)*A10-A9*G*G
E78=2.*A*A12-A11*(F+GA)-A12*G*G
E79=2.*A11*A+(F+GA)*A12-A11*G*G
E80=2.*A*A14-A13*(FA-EA)-A14*G*G
E81=2.*A*A13+(FA-EA)*A14-A13*G*G
E82=2.*A*A16-A15*(FA+EA)-A16*G*G
E83=2.*A*A15+(FA+EA)*A16-A15*G*G
E84=2.*A*A18-A17*(FA-GA)-A18*G*G
E85=2.*A*A17+(FA-GA)*A18-A17*G*G
E86=2.*A*A20-A19*(FA+GA)-A20*G*G
E87=2.*A*A19+(FA+GA)*A20-A19*G*G
E87P=2.*A*A21
E88=2.*A*A23-2.*EA*A22
E89=2.*A*A22+2.*EA*A23
E90=2.*A*A25-A24*(GA-EA)
E91=2.*A*A24+(GA-EA)*A25
E92=2.*A*A27-A26*(GA+EA)
E93=2.*A*A26+(GA+EA)*A27
E94=2.*A*A29-2.*GA*A28
E95=2.*A*A28+2.*GA*A29
E96=2.*A*A30
E97=2.*A*A32-2.*A31*F
E98=2.*A*A31+2.*F*A32
E99=A33*(-4.*G*G+2.*A)
E100=2.*A*A35-2.*A34*F-4.*A35*G*G
E101=2.*A*A34+2.*A35*F-4.*A34*G*G
E102=2.*A*A37-A36*(FA-F)
E103=2.*A*A36+A37*(FA-F)
E104=2.*A*A39-A38*(FA+F)
E105=2.*A*A38+A39*(FA+F)
E106=2.*A*A41-A40*(FA-F)-4.*A41*G*G
E107=2.*A*A40+A41*(FA-F)-4.*A40*G*G
E108=2.*A*A43-A42*(FA+F)-4.*A43*G*G
E109=2.*A42*A+A43*(FA+F)-4.*A42*G*G
E110=2.*A*A44
E111=2.*A*A46-2.*A45*FA
E112=2.*A45*A+2.*A46*FA
E113=(-4.*G*G+2.*A)*A47
E114=2.*A*A49-2.*A48*FA-4.*A49*G*G
E115=2.*A*A48+2.*A49*FA-4.*A48*G*G
E116=RATIO*(2.+SG)*FO
E117=(AC*(1.+A)-AD*EA)*A1H
E118=(AD*(1.+A)+AC*EA)*A1H
```

```
E119=(ACP*(1.+A)-ADP*GA)*A2H
E120=(ADP*(1.+A)+ACP*GA)*A2H
DB=EXP(2.*(A-1.)*C)-1.
DC=EXP(2.*A*C)-1.
DF=EXP(2.*(2.*A-1.)*C)-1.
DG=1.+EXP((3.*A-1.)*C)
DD=DG
DE=DG
G1=DB/4./(A-1.)
G2=-DB/4./F
G3=(A-1.)*DB*(1./(FA-F)**2-1./(FA+F)**2)
G4=-DB/2.*(1./(F+FA)-1./(FA-F))
G5=.5*(-DB)*(1./(FA+F)+1./(FA-F))
G6=-DB/4./FA
G7=DC/4./A
G8=-DC/4./F
G9=A*DC*(1./(FA-F)**2-1./(FA+F)**2)
G10=.5*DC*(-1./(F+FA)+1./(FA-F))
G11=DD*(1./2./EA**2-1./2./(2.*F-EA)**2)*(3.*A-1.)*(-1.)
G12=DD*(1./2./(2.*F-EA)+1./2./EA)
G13=(3.*A-1.)*(-DD)*(1./2./EA**2-1./2./(2.*F+EA)**2)
G14=DD*(1./2./(2.*F+EA)-1./2./EA)
G15=-DD*(3.*A-1.)*(1./2./GA**2-1./2./(2.*F-GA)**2)
G16=DD*(1./2./(2.*F-GA)+1./2./GA)
G17=DD*(3.*A-1.)*(1./2./GA**2-1./2./(2.*F+GA)**2)
G18=DD*(1./2./(2.*F+GA)-1./2./GA)
G19=-DD*(3.*A-1.)*(1./2./(F-FA+EA)**2-1./2./(F+FA-EA)**2)
G20=-DD*(1./2./(F+FA-EA)-1./2./(FA-F-EA))
G21=DD*(3.*A-1.)*(1./2./(F-FA-EA)**2-1./2./(F+FA+EA)**2)
G22=DD*(1./2./(F+FA+EA)-1./2./(FA-F+EA))
G23=-DD*(3.*A-1.)*(1./2./(F-FA+GA)**2-1./2./(F+FA-GA)**2)
G24=DD*(1./2./(F+FA-GA)-1./2./(FA-F-GA))
G25=-DD*(3.*A-1.)*(1./2./(F-FA-GA)**2-1./2./(F+FA+GA)**2)
G26=DD*(1./2./(F+FA+GA)-1./2./(FA-F+GA))
G27=.5*(-DC)*(1./(F+FA)+1./(FA-F))
G28=-DC/4./FA
G29=-DD*(3.*A-1.)*(1./2./(FA-F+EA)**2-1./2./(FA+F-EA)**2)
G30=DD*(1./2./(FA+F-EA)+1./2./(FA-F+EA))
G31=-DD*(3.*A-1.)*(1./2./(FA-F-EA)**2-1./2./(FA+F+EA)**2)
G32=DD*(1./2./(FA+F+EA)+1./2./(FA-F-EA))
G33=-DD*(3.*A-1.)*(1./2./(FA-F+GA)**2-1./2./(FA+F-GA)**2)
G34=DD*(1./2./(FA+F-GA)+1./2./(FA-F+GA))
G35=(3.*A-1.)*DD*(-1.)*(1./2./(FA-F-GA)**2-1./2./(FA+F+GA)**2)
G36=DD*(1./2./(FA+F+EA)+1./2./(FA-F-EA))
G37=-DD*(3.*A-1.)*(1./2./EA**2-1./(2.*FA-EA)**2)
G38=DD*(1./2./(2.*FA-EA)+1./2./EA)
G39=-DD*(3.*A-1.)*(1./2./FA**2-1./(2.*FA+EA)**2)
G40=DD*(1./2./(2.*FA+EA)-1./2./EA)
G41=(3.*A-1.)*DD*(1./2./GA**2-1./(2.*FA-GA)**2)
G42=DD*(1./2./(2.*FA-GA)+1./2./GA)
G43=-DD*(3.*A-1.)*(1./2./GA**2-1./(2.*FA+GA)**2)
```

```
 G44=DD*(1./2./(2.*FA+GA)-1./2./GA)
 G45=.25*DE*(1./EA-1./(2.*F+EA)+1./(2.*F-EA))
 G45P=(3.*A-1.)/4.*DE*(1./(2.*F+EA)**2-1./(2.*F-EA)**2    )
 G46=DE/4.*(1./(FA-F+EA)-1./(FA-F-EA)-1./(F+FA+EA)+1./(F+FA-EA))
 G47=(3.*A-1.)/4.*(-DE)*(1./(F+FA-EA)**2-1./(F+FA+EA)**2-1./(FA-F
1-EA)**2+1./(FA-F+EA)**2)
 G48=DF/4.*(-1./2./EA+1./2./F-1./2./(F-EA))
 G49=(2.*A-1.)/2.*DF*(1./4./(F-EA)**2-1./4./F**2+1./4./(2.*A-1.)**2
1-1./4./EA**2)
 G50=-DF/4.*(1./2./EA+1./2./F-1./2./(F+EA))
 G51=(2.*A-1.)/2.*DF*(1./4./F**2-1./4./(F+EA)**2-1./4./(2.*A-1.)**2
1+1./4./EA**2)
 G52=-DF/4.*(1./(2.*F-EA-GA)-1./(GA-EA)           -1./(2.*F-GA+EA)
1+1./(EA+GA))
 G53=(2.*A-1.)/2.*DF*(1./(GA-EA)**2+1./(2.*F-EA-GA)**2-1./(EA+GA)
1**2-1./(2.*F+EA-GA)**2)
 G54=-DF/4.*(1./(2.*F-EA+GA)+1./(GA+EA)-1./(2.*F+GA+EA)+1./(EA-GA))
 G55=(2.*A-1.)/2.*DF*(1./(GA+EA)**2-1./(2.*F-EA+GA)**2-1./(EA
1-GA)**2+1./(2.*F+EA+GA)**2)
 G56=-DF/4.*(1./(FA-F)+1./(FA+F-2.*EA)-1./(F+FA)-1./(FA-F-2.*EA))
 G57=(2.*A-1.)/2.*DF*(1./(F+FA-2.*EA)**2-1./(F+FA)**2-1./(FA-F-
12.*EA)**2+1./(FA-F)**2)
 G58=-DF/4.*(1./(FA-F+2.*EA)-1./(FA-F)-1./(FA+F+2.*EA)+1./(FA+F))
 G59=(2.*A-1.)/2.*DF*(1./(F+FA)**2-1./(F+FA+2.*EA)**2-1./(FA-F)**2+
11./(FA-F+2.*EA)**2)
 G60=-DF/4.*(1./(FA+F-EA-GA)+1./(FA-F-GA+EA)-1./(FA+F-GA+EA)
1-1./(FA-F-EA-GA))
 G61=(2.*A-1.)/2.*DF*(1./(FA+F-EA-GA)**2+1./(FA-F -GA+EA)**2
1-1./(F+FA+EA-GA)**2-1./(FA-F-EA-GA)**2)
 G62=-DF/4.*(1./(FA+F-EA+GA)+1./(FA-F+GA+EA)-1./(FA+F+GA+EA)
1-1./(FA-F-EA+GA))
 G63=(2.*A-1.)/2.*DF*(1./(FA+F-EA+GA)**2+1./(FA-F+GA+EA)**2
1-1./(FA+F+GA+EA)**2-1./(FA-F-EA+GA)**2)
 G64=(3.*A-1.)/4.*DE*(1./(2.*F+EA)**2+1./(2.*F-EA)**2-2./F**2)
 G65=DE/4.*(1./(2.*F+EA)+1./(2.*F-EA))
 G66=(3.*A-1.)/4.*(-DE)*(1./(FA-F-EA)**2-1./(FA+F+EA)**2+1./
1(FA-F+EA)**2-1./(F+FA-EA)**2)
 G67=DE/4.*(1./(F+FA+EA)+1./(FA+F-EA)-1./(FA-F+EA)-1./(FA-F-EA))
 G68=(2.*A-1.)/2.*DF*(1./4./EA**2+1./4./(2.*A-1.)**2-1./4./(
1F-EA)**2-1./4./F**2)
 G69=-DF/4.*(1./2./F+1./2./(F-EA)+1./2./EA)
 G70=(2.*A-1.)/2.*DF*(1./4./EA**2+1./4./(2.*A-1.)**2-1./4.
1/(F+EA)**2-1./4./F**2)
 G71=-DF/4.*(1./2./(F+EA)+1./2./F-1./2./EA)
 G72=(2.*A-1.)/2.*DF*(1./(EA+GA)**2-1./(2.*F-GA+EA)**2+1./(GA-
1EA)**2-1./(2.*F-EA-GA)**2)
 G73=-DF/4.*(1./(2.*F+EA-GA)+1./(EA+GA)+1./(2.*F-EA-GA)+1./(GA-EA))
 G74=(2.*A-1.)/2.*DF*(1./(EA-GA)**2-1./(2.*F+GA+EA)**2+1./(GA
1+EA)**2-1./(2.*F-EA+GA)**2)
 G75=-DF/4.*(1./(2.*F+EA+GA)+1./(EA-GA)+1./(2.*F-EA+GA)-1./(GA+EA))
 G76=(2.*A-1.)/2.*DF*(1./(FA-F-2.*EA)**2-1./(FA+F)**2+1./(FA-F)**2
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1-1./(FA+F-2.*EA)**2)
 G77=-DF/4.*(1./(FA+F)-1./(FA-F-2.*EA)+1./(FA+F-2.*EA)-1./(FA-F))
 G78=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(FA+F+2.*EA)**2+1./(FA-F
1+2.*EA)**2-1./(FA+F)**2)
 G79=-DF/4.*(1./(FA+F+2.*EA)-1./(FA-F)+1./(FA+F)-1./(FA-F+2.*EA))
 G80=(2.*A-1.)/2.*DF*(1./(FA-F-GA-EA)**2-1./(FA+F-GA+EA)**2
1+1./(FA-F-GA+EA)**2-1./(FA+F-EA-GA)**2)
 G81=-DF/4.*(1./(FA+F+EA-GA)-1./(FA-F-GA-EA)+1./(FA+F-GA-EA)
1-1./(FA-F-GA+EA))
 G82=(2.*A-1.)/2.*DF*(1./(FA-F+GA-EA)**2-1./(FA+F+GA+EA)**2
1+1./(FA-F+GA+EA)**2-1./(FA+F-EA+GA)**2)
 G83=-DF/4.*(1./(FA+F+EA+GA)-1./(FA-F+GA-EA)+1./(FA+F-GA-EA)
1-1./(FA-F+GA+EA))
 G84=DE/4.*(1./GA-1./(2.*F+GA)+1./(2.*F-GA))
 G85=(2.*A-1.)/2.*DE*(1./(2.*F+GA)**2-1./(2.*F-GA)**2)
 G86=DE/4.*(1./(FA-F+GA)-1./(FA-F-GA)-1./(F+FA+GA)+1./(F+FA-GA))
 G87=-DE*(3.*A-1.)/4.*(1./(F+FA-GA)**2-1./(F+FA+GA)**2-1./
1(FA-F-GA)**2+1./(FA-F+GA)**2)
 G88=-DF/4.*(1./(2.*F-EA-GA)+1./(GA-EA)-1./(2.*F-EA+GA)+1./(EA+GA))
 G89=(2.*A-1.)/2.*DF*(1./(GA-EA)**2+1./(2.*F-EA-GA)**2-1./(EA
1+GA)**2-1./(2.*F+GA-EA)**2)
 G90=-DF/4.*(1./(2.*F-GA+EA)+1./(GA+EA)-1./(2.*F+EA+GA)+1./(GA-EA))
 G91=(2.*A-1.)/2.*DF*(1./(GA+EA)**2+1./(2.*F+EA-GA)**2-1./(GA-
1EA)**2-1./(2.*F+GA+EA)**2)
 G92=DF/4.*(-1./2./GA+1./2./F-1./2./(F-GA))
 G93=(2.*A-1.)/2.*DF*(1./4./(F-GA)**2-1./4./F**2+1./4./(2.*A-1.)
1**2-1./4./GA**2)
 G94=-DF/4.*(1./2./GA+1./2./F-1./2./(F-EA))
 G95=(2.*A-1.)/2.*DF*(1./4./F**2-1./4./(F+GA)**2-1./4./(2.*A-1.)**2
1+1./4./GA**2)
 G96=-DF/4.*(1./(FA-EA+F-GA)+1./(FA-F-EA+GA)-1./(FA+F-EA+GA)
1-1./(FA-F-GA-EA))
 G97=(2.*A-1.)/2.*DF*(1./(FA+F-EA-GA)**2+1./(FA-F-EA+GA)**2
1-1./(F+GA+FA-EA)**2 -1./(FA-F-EA-GA)**2)
 G98=-DF/4.*(1./(FA+F+EA-GA)+1./(FA-F+EA+GA)-1./(FA+F+EA+GA)-1.
1/(FA-F-GA+EA))
 G99=(2.*A-1.)/2.*DF*(1./(FA+F+EA-GA)**2+1./(FA-F+EA+GA)**2-1./
1(F+FA+EA+GA)**2-1./(FA-F+EA-GA)**2)
 G100=-DF/4.*(1./(FA-F)+1./(FA+F-2.*GA)-1./(F+FA)+1./(F-FA+2.*GA))
 G101=(2.*A-1.)/2.*DF*(1./(F+FA-2.*GA)**2-1./(F+FA)**2-1./(FA-F
1-2.*GA)**2+1./(FA-F)**2)
 G103=(2.*A-1.)/2.*DF*(1./(F+FA)**2-1./(F+FA+2.*GA)**2-1./(FA
1-F)**2+1./(FA-F+2.*GA)**2)
 G102=-DF/4.*(1./(FA-F+2.*GA)-1./(FA-F)-1./(FA+F+2.*GA)+1./(FA+F))
 G104=(3.*A-1.)/4.*DE*(1./(2.*F+GA)**2+1./(2.*F-GA)**2-2./F**2)
 G105=DE/4.*(1./(2.*F+GA)+1./(2.*F-GA))
 G106=(3.*A-1.)/4.*(-DE)*(1./(FA-F-GA)**2-1./(FA+F+GA)**2-1./(F+
1FA-GA)**2+1./(F-FA-GA)**2)
 G107=DE/4.*(1./(F+FA+GA)+1./(FA+F-GA)-1./(FA-F+GA)-1./(FA-F-GA))
 G108=(2.*A-1.)/2.*DF*(1./(EA+GA)**2-1./(2.*F-EA+GA)**2+1./(GA-EA)
1**2-1./(2.*F-EA-GA)**2)
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 G109=-DF/4.*(1./(2.*F-EA+GA) +1./(EA+GA)+1./(2.*F-EA-GA)+1./(EA-
1GA))
 G110=(2.*A-1.)/2.*DF*(1./(EA-GA)**2-1./(2.*F+ EA+GA)**2+1./(EA+
1GA)**2-1./(2.*F-GA+EA)**2)
 G111=-DF/4.*(1./(2.*F+EA+GA)+1./(GA-EA)+1./(2.*F-GA+EA)-1./(EA+
1GA))
 G112=(2.*A-1.)/2.*DF*(1./4./GA**2+1./4./(2.*A-1.)**2-1./4./(F-GA)
1**2-1./4./F**2)
 G113=-DF/4.*(1./2./F+1./2./(F-GA)+1./2./GA)
 G114=(2.*A-1.)/2.*DF*(1./4./GA**2+1./4./(2.*A-1.)**2-1./4./(F+GA
1)**2-1./4./F**2)
 G115=-DF/4.*(1./2./(F+GA)+1./2./F-1./2./GA)
 G116=(2.*A-1.)/2.*DF*(1./(FA-F-EA-GA)**2-1./(FA+F-EA+GA)**2
1+1./(FA-F-EA+GA)**2-1./(FA+F-EA-GA)**2)
 G117=-DF/4.*(1./(FA+F+GA-EA)-1./(FA-F-EA-GA)+1./(FA+F-EA-GA)
1-1./(FA-F-EA+GA))
 G118=(2.*A-1.)/2.*DF*(1./(FA-F+EA-GA)**2-1./(FA+F+EA+GA)**2
1+1./(FA-F+EA+GA)**2-1./(FA+F-GA+EA)**2)
 G119=-DF/4.*(1./( FA+F+EA+GA)-1./(FA-F+EA-GA)+1./(FA+F+EA-GA)
1-1./(FA-F+EA+GA))
 G120=(2.*A-1.)/2.*DF*(1./(FA-F-2.*GA)**2-1./(FA+F)**2+1./
1(FA-F)**2-1./(FA+F-2.*GA)**2)
 G121=-DF/4.*(1./(FA+F)-1./(FA-F-2.*GA)+1./(FA+F-2.*GA)-1./(FA-F))
 G122=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(FA+F+2.*GA)**2+1./(FA-F
1+2.*GA)**2-1./(FA+F)**2)
 G123=-DF/4.*(1./(FA+F+2.*GA)-1./(FA-F)+1./(FA+F)-1./(FA-F+2.*GA))
 G124=DF/4./(2.*A-1.)
 G125=-DF/2.*(1./2./EA+1./2./(2.*F-2.*EA)-1./2./(2.*F+2.*EA))
 G126=(2.*A-1.)*DF*(1./4./EA**2-1./2./(2.*F+2.*EA)**2-1./2./
1(2.*F-2.*EA)**2)
 G127=-DF/2.*(1./(GA-EA)+1./2./(2.*F-GA+EA)-1./2./(2.*F+GA-EA))
 G128=(2.*A-1.)*DF*(1./(GA-EA)**2-1./2./(2.*F+GA-EA)**2-1./2./
1(2.*F-GA+EA)**2)
 G129=-DF/2.*(1./(EA+GA)+1./2./(2.*F-EA-GA)-1./2./(2.*F+EA+GA))
 G130=(2.*A-1.)*DF*(1./(EA+GA)**2-1./2./(2.*F+EA+GA)**2-1./
12./(2.*F-EA-GA)**2)
 G131=-DF/2.*(1./2./GA+1./2./(2.*F-2.*GA)-1./2./(2.*F+2.*GA))
 G132=(2.*A-1.)*DF*(1./4./GA**2-1./2./(2.*F+2.*GA)**2-1./
12./(2.*F-2.*GA)**2)
 G133=3./16./F*(-DF)
 G134=-DF/8./(2.*A-1.)
 G135=-DF/4.*(1./(FA-F)-(FA-3.*F)/(4.*(2.*A-1.)**2+(FA-3.*F)**2))
 G136=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(4.*(2.*A-1.)**2+(FA-3.*F)
1**2))
 G137=-DF/4.*(1./2./(FA+F)-1./(FA-F)-1./(3.*F+FA))
 G138=(2.*A-1.)/2.*DF*(1./(FA+F)**2-1./(3.*F+FA)**2-1./(FA
1-F)**2)
 G139=-DF/2.*(1./2./FA+1./2./(2.*F-2.*FA)-1./2./(2.*F+2.*FA))
 G140=(2.*A-1.)*(1./4./FA**2-1./2./(2.*F+2.*FA)**2-1./2./(2.*F
1-2.*FA)**2)
 G141=(EXP(2.*(A-1.)*C)-1.)/4./(A-1.)
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 G142=(EXP(2.*A*C)-1.)/4./A
 G143=EXP(2.*A*C)/2.*(C/2./A-1./4./A**2)+1./8./A**2-A*C*EXP(2.*A*C)
1     /4./F**2-1./8./F**2*(EXP(2.*A*C)-1.)
 G144=(EXP((2.*A+SG)*C)-1.)/2./(2.*A+SG)
 G145=DG/4.*(2./EA+1./(2.*F-EA)-1./(2.*F+EA))
 G146=(3.*A-1.)/4.*DG*(1./(2.*F+EA)**2+1./(2.*F-EA)**2-2./EA**2)
 G147=DG/4.*(2./GA  +1./(2.*F-GA)-1./(2.*F+GA))
 G148=DG/4.*(1./(2.*F+GA)**2+1./(2.*F-GA)**2-2./GA**2)  *(3.*A-1.)
 G149=-DF/4./F
 G150=(2.*A-1.)/2.*DF*(1./(2.*F-2.*EA)**2-1./(2.*F+2.*EA)**2)
 G151=-DF/4.*(1./2./(F+EA)+1./2./(F-EA))
 G152=(2.*A-1.)/2.*DF*(1./(2.*F-GA+EA)**2-1./(2.*F+GA-EA)**2)
 G153=-DF/4.*(1./(2.*F+GA-EA)+1./(2.*F-GA+EA))
 G154=(2.*A-1.)/2.*DF*(1./(2.*F-GA-EA)**2-1./(2.*F+GA+EA)**2)
 G155=-DF/4.*(1./(2.*F+GA+EA)+1./(2.*F-EA-GA))
 G156=(2.*A-1.)/2.*DF*(1./ 4./(F-GA)**2-1./4./(F+GA)**2)
 G157=-DF/4.*(1./2./(F+GA)+1./2./(F-GA))
 G158=DF/8./(2.*A-1.)
 G159=-DF/16./F
 G160=(2.*A-1.)/2.*DF*(1./(4.*(2.*A-1.)**2+(FA-3.*F)**2)-1./
1(FA+F)**2)
 G161=-DF/4.*(1./(FA+F)-1./(4.*(2.*A-1.)**2+(FA-3.*F)**2)*
1(FA-3.*F))
 G162=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(FA+3.*F)**2)
 G163=-DF/4.*(1./(FA+3.*F)-1./(FA-F))
 G164=(2.*A-1.)/2.*DF*(1./(2.*FA-2.*F)**2-1./(2.*FA+2.*F)**2)
 G165=-DF/4.*(1./(2.*F+2.*FA)-1./2./(FA-F))
 G166=(1.-EXP(2.*(A-1.)*C))/4./F
 G167=(1.-EXP(2.*A*C))/4./F
 G168=-C*EXP(2.*A*C)/4./F+A/4./F**3*(EXP(2.*A*C)-1.)
 G169=(1.-EXP((2.*A+SG)*C))/4./F
 G170=(3.*A-1.)/4.*(-DG)*(1./(2.*F-EA)**2-1./(2.*F+EA)**2)
 G171=DG/4.*(1./(2.*F+EA)+1./(2.*F-EA))
 G172=(3.*A-1.)/4.*DG*(-1./(2.*F-GA)**2+1./(2.*F+GA)**2)
 G173=DG/4.*(1./2.*F+GA)+1./(2.*F-GA))
 G174=(2.*A-1.)*DF*(1./(F-FA)**2-1./(F+FA)**2)
 G175=-DF/4.*(1./(FA-F+2.*EA)-1./(FA-F-2.*EA)-1./(FA+F+2.*EA)
1+1./(FA+F-2.*EA))
 G176=(2.*A-1.)/2.*DF*(1./(FA-F+2.*EA)**2+1./(FA-F-2.*EA)**2
1-1./(FA+F+2.*EA)**2-1./(FA+F-2.*EA)**2)
 G177=-DF/4.*(1./(FA-F+GA-EA)-1./(FA-F-GA+EA)-1./(FA+F+GA-EA)
1+1./(FA+F-GA+EA))
 G178=(2.*A-1.)/2.*DF*(1./(FA-F+GA-EA)**2+1./(FA-F-GA+EA)**2-1.
1/(FA+F+GA-EA)**2-1./(FA+F-GA+EA)**2)
 G179=-DF/4.*(1./(FA-F+GA+EA)-1./(FA-F-GA-EA)-1./(FA+F+GA+EA)
1+1./(FA+F-GA-EA))
 G180=(2.*A-1.)/2.*DF*(1./(FA-F+GA+EA)**2+1./(FA-F-GA-EA)**2
1-1./(FA+F+GA+EA)**2-1./(FA+F-EA-GA)**2)
 G181=-DF/4.*(1./(FA-F+2.*GA)-1./(FA-F-2.*GA)-1./(FA+F+2.*GA)
1+1./(FA+F-2.*GA))
 G182=(2.*A-1.)/2.*DF*(1./(FA-F+2.*GA)**2+1./(FA-F-2.*GA)**2
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1-1./(FA+F+2.*GA)**2-1./(FA+F-2.*GA)**2)
 G183=-DF/4.*(1./(F+FA)-1./(F-FA)-1./(3.*F+FA)+(3.*F-FA)/(4.*
1(2.*A-1.)**2+(3.*F-FA)**2))
 G184=(2.*A-1.)/2.*DF*(1./((3.*F-FA)**2+4.*(2.*A-1.)**2)-1./
1(3.*F+FA)**2-1./(F-FA)**2+1./(F+FA)**2)
 G185=-DF/4.*(1./2./(FA-F)-1./2./FA+1./2./F)
 G186=(2.*A-1.)/2.*(1./4./(FA-F)**2+1./4./(2.*A-1.)**2-1./4./FA**2
1-1./4./F**2)
 G187=-DF/4.*(1./2./FA+1./2./F-1./2./(FA+F))
 G188=(2.*A-1.)/2.*DF*(1./4./FA**2+1./4./F**2-1./4./(2.*A-1.)**2
1-1./4./(FA+F)**2)
 G189=-DF/4.*(1./(3.*FA-F)+1./(FA+F)-1./(3.*FA+F)-1./(FA-F))
 G190=(2.*A-1.)/2.*DF*(1./(3.*FA-F)**2+1./(FA+F)**2-1./(3.*FA
1+F)**2-1./(FA-F)**2)
 G191=(A-1.)*(EXP(2.*(A-1.)*C)-1.)*(1./(FA-F)**2-1./(FA+F)**2)
 G192=A*(EXP(2.*A*C)-1.)*(1./(FA-F)**2-1./(FA+F)**2)
 G193=-A*C*EXP(2.*A*C)/(FA+F)**2+A*C*EXP(2.*A*C)/(FA-F)**2-(1./
12./(FA-F)**2-1./2./(FA+F)**2)*(EXP(2.*A*C)-1.)
 G194=(2.*A+SG)/2.*(EXP((2.*A+SG)*C)-1.)*(1./(FA-F)**2-1./(FA+F)
1**2)
 G195=-DG/4.*(1./(FA+F+EA)+1./(FA-F-EA)-1./(FA-F+EA)+1./(FA+F-EA))
 G196=(3.*A-1.)*(-DG)*(1./(FA-F+EA)**2+1./(FA-F-EA)**2-1./
1(FA+F+EA)**2-1./(FA+F-EA)**2)/4.
 G198=DG/4.*(3.*A-1.)*(1./(FA-F+EA)**2+1./(FA-F-GA)**2-1./
1(FA+F+GA)**2-1./(FA+F-GA)**2)
 G197=DG/4.*(1./(FA-F+GA)-1./(FA-F-GA)-1./(FA+F+GA)+1./(FA+F-GA))
 G199=-DF/2.*(1./(F+FA)-1./(FA-F))
 G200=(2.*A-1.)/2.*DF*(1./(FA+F-2.*EA)**2-1./(FA+F+2.*EA)**2
1-1./(FA-F-2.*EA)**2+1./(FA-F+2.*EA)**2)
 G201=-DF/4.*(1./(FA+F+2.*EA)+1./(F+FA-2.*EA)-1./(FA-F+2.*EA)
1-1./(FA-F-2.*EA))
 G202=(2.*A-1.)/2.*DF*(1./(FA+F-GA+EA)**2-1./(FA+F+GA-EA)**2
1-1./(FA-F-GA+EA)**2+1./(FA-F+GA-EA)**2)
 G203=-DF/4.*(1./(FA+F+GA-EA)+1./(FA+F-GA+EA)-1./(FA-F+GA-EA)
1-1./(FA-F-GA+EA))
 G204=(2.*A-1.)/2.*DF*(1./(FA+F-GA-EA)**2-1./(FA+F+GA+EA)**2
1-1./(FA-F-GA-EA)**2+1./(FA-F+GA+EA)**2)
 G205=-DF/4.*(1./(FA+F+GA+EA)+1./(F+FA-GA-EA)-1./(FA-F+GA+EA)
1-1./(FA-F-GA-EA))
 G206=(2.*A-1.)/2.*DF*(1./(FA+F-2.*GA)**2-1./(FA+F+2.*GA)**2
1-1./(FA-F-2.*GA)**2+1./(FA-F+2.*GA)**2)
 G207=-DF/4.*(1./(FA+F+2.*GA)+1./(F+FA-2.*GA)-1./(FA-F+2.*GA)
1-1./(FA-F-2.*GA))
 G208=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(FA+3.*F)**2-1./((FA-3.*F)
1**2+4.*(2.*A-1.)**2)+1./(FA+F)**2)
 G209=-DF/4.*(1./(FA+3.*F)+1./(FA-F)-1./(FA+F)-(FA-3.*F)/
1((FA-3.*F)**2+4.*(2.*A-1.)**2))
 G210=(2.*A-1.)/2.*DF*(1./4./F**2-1./4./FA**2-1./4./(2.*A-1.)**2
1+1./4./(FA-F)**2)
 G211=-DF/4.*(1./2./FA+1./2./F-1./2./(FA-F))
 G212=(2.*A-1.)/2.*DF*(1./4./(2.*A-1.)**2-1./4./(FA+F)**2-1./4./
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1F**2+1./4./FA**2)
 G213=-DF/4.*(1./2./(F+FA)-1./2./FA+1./2./F)
 G214=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(3.*FA+F)**2-1./
1(FA+F)**2+1./(3.*FA-F)**2)
 G215=-DF/4.*(1./(3.*FA+F)-1./(FA-F)-1./(3.*FA-F)+1./(FA+F))
 G216=(1.-EXP(2.*(A-1.)*C))*(1./(F+FA)-1./(FA-F))/2.
 G217=(1.-EXP(2.*A*C))/2.*(1./(F+FA)-1./(FA-F))
 G218=C/2.*EXP(2.*A*C)*(1./(FA-F)-1./(FA+F))+(EXP(2.*A*C)-1.)/2.
1*(1./(FA+F)**3-1./(FA-F)**3)
 G219=(1.-EXP((2.*A+SG)*C))/2.*(1./(F+FA)-1./(FA-F))
 G220=(3.*A-1.)/4.*DG*(-1./(FA+F-EA)**2+1./(FA+F+EA)**2+1./
1(FA-F-EA)**2-1./(FA-F+EA)**2)
 G221=DG/4.*(1./(FA+F+EA)+1./(F+FA-EA)-1./(FA-F+EA)-1./(FA-F-EA))
 G222=(3.*A-1.)/4.*DG*(-1./(FA+F-GA)**2+1./(FA+F+GA)**2+1./(FA-F
1-GA)**2-1./(FA-F+GA)**2)
 G223=DG/4.*(1./(FA+F+GA)+1./(F+FA-GA)-1./(FA-F+GA)-1./(FA-F-GA))
 G224=-DG*(3.*A-1.)/4.*(1./(FA-EA+F)**2+1./(FA-EA-F)**2-1./
1(FA+F+EA)**2-1./(FA+EA-F)**2)
 G225=DE/4.*(-1./(2.*FA+EA)+1./(2.*FA-EA )+2./EA)
 G226=-DE*(3.*A-1.)/4.*(1./(2.*FA+EA)**2-1./(2.*FA-EA)**2)
 G227=-DF/4.*(1./(FA+F-2.*EA)-1./(FA-F)-1./(FA+F)+1./(FA-F+2.*EA))
 G228=(2.*A-1.)/2.*DF*(1./(FA+F-2.*EA)**2+1./(FA-F)**2
1-1./(FA+F)**2-1./(FA-F+2.*EA)**2)
 G229=-DF/4.*(1./(FA+F)-1./(FA-F-2.*EA)-1./(FA+F+2.*EA)+1./(FA-F))
 G230=(2.*A-1.)/2.*DF*(1./(FA+F)**2+1./(FA-F-2.*EA)**2
1-1./(FA+F+2.*EA)**2-1./(FA-F)**2)
 G231=-DF/4.*(1./(FA+F-EA-GA)-1./(FA-F-EA+GA)-1./(FA+F+EA-GA)
1+1./(FA-F+EA+GA))
 G232=(2.*A-1.)/2.*DF*(1./(FA+F-EA-GA)**2+1./(FA-F-EA+GA)**2
1-1./(FA+F+EA-GA)**2-1./(FA-F+EA+GA)**2)
 G233=-DF/4.*(1./(FA+F-EA+GA)-1./(FA-F-EA-GA)-1./(FA+F+EA+GA)
1+1./(FA-F+EA-GA))
 G234=(2.*A-1.)/2.*DF*(1./(FA+F-EA+GA)**2+1./(FA-F-EA-GA)**2-1./
1(FA+F+EA+GA)**2-1./(FA-F+EA-GA)**2)
 G235=-DF/4.*(1./2./(FA-EA)-1./2./FA+1./2./EA)
 G236=(2.*A-1.)/2.*DF*(1./4./(2.*A-1.)**2+1./4./(FA-EA)**2
1-1./4./FA**2-1./4./EA**2)
 G237=-DF/4.*(1./2./FA+1./2./EA-1./2./(FA+EA))
 G238=(2.*A-1.)/2.*DF*(1./4./FA**2+1./4./EA**2-1./4./(2.*A-1.)**2
1-1./4./(FA+EA)**2)
 G239=-DF/4.*(1./(2.*FA-EA-GA)-1./(GA-EA)-1./(2.*FA+EA-GA)
1+1./(EA+GA))
 G240=(2.*A-1.)/2.*DF*(1./(2.*FA-EA-GA)**2+1./(GA-EA)**2
1-1./(2.*FA+EA-GA)**2-1./(EA+GA)**2)
 G241=-DF/4.*(1./(2.*FA-EA+GA)+1./(GA+EA)-1./(2.*FA+EA+GA)
1+1./(EA-GA))
 G242=(2.*A-1.)/2.*(1./(2.*FA-EA+GA)**2+1./(GA+EA)**2-1./(2.*FA
1+EA+GA)**2-1./(EA-GA)**2) *DF
 G243=DE/4.*(1./(FA+F+EA)+1./(FA+EA-F)+1./(FA+F-EA)+1./(FA-F-EA))
 G244=(3.*A-1.)/4.*(-2./EA**2+1./(2.*FA+EA)**2+1./(2.*FA-EA)**2)*DE
 G245=DE/4.*(1./(2.*FA+EA) +1./(2.*FA-EA))
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 G246=(2.*A-1.)/2.*DF*(1./(FA-F+2.*EA)**2-1./(FA+F)**2
1+1./(FA-F)**2-1./(FA+F-2.*EA)**2)
 G247=DF/4.*(1./(FA+F)+1./(FA-F+2.*EA)+1./(FA+F-2.*EA)+1./(FA-F))
 G248=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(FA+F-2.*EA)**2
1+1./(FA-F-2.*EA)**2-1./(FA+F)**2)
 G249=-DF/4.*(1./(FA+F+2.*EA)+1./(FA-F)+1./(FA+F)+1./(FA-F-2.*EA))
 G250=(2.*A-1.)/2.*DF*(1./(FA-F+EA+GA)**2-1./(FA+F+EA-GA)**2
1+1./(FA-F-EA+GA)**2-1./(FA+F-EA-GA)**2)
 G251=-DF/4.*(1./(FA+F+EA-GA)+1./(FA+EA-F +GA)+1./(FA+F-EA-GA)
1+1./(FA-F-EA+GA))
 G252=(2.*A-1.)/2.*DF*(1./(FA-F+EA-GA)**2-1./(FA+F+EA+GA)**2
1+1./(FA-F-EA-GA)**2-1./(FA+F-EA+GA)**2)
 G253=-DF/4.*(1./(FA+F+EA+GA)+1./(FA-F+EA-GA)+1./(FA+F -EA +GA)
1+1./(FA-F-EA-GA))
 G254=(2.*A-1.)/2.*DF*(1./4./EA**2-1./4./FA**2+1./4./(2.*A-1.)**2
1-1./4./(FA-EA)**2)
 G255=-DF/4.*(1./2./FA+1./2./EA+1./2./(FA-EA))
 G256=(2.*A-1.)/2.*DF*(1./4./(2.*A-1.)**2-1./4./(FA+EA)**2
1+1./4./EA**2-1./4./FA**2)
 G257=-DF/4.*(1./2./(FA+EA)+1./2./FA-1./2./EA)
 G258=(2.*A-1.)/2.*DF*(1./(EA+GA)**2-1./(2.*FA+EA-GA)**2
1+1./(GA-EA)**2-1./(2.*FA-EA-GA)**2)
 G259=-DF/4.*(1./(2.*FA+EA-GA)+1./(EA+GA)+1./(2.*FA-GA-EA)+1./(GA-
1EA))
 G260=(2.*A-1.)/2.*DF*(1./(GA-EA)**2-1./(2.*FA+GA+EA)**2
1+1./(GA+EA)**2-1./(2.*FA-EA+GA)**2)
 G261=-DF/4.*(1./(2.*FA+EA+GA)-1./(GA-EA)+1./(2.*FA-EA+GA)
1-1./(GA+EA))
 G262=(3.*A-1.)/4.*(-1./(FA+F-GA)**2-1./(FA-F-GA)**2
1+1./(FA+F+GA)**2+1./(FA+GA-F)**2)*DG
 G263=DG/4.*(1./(2.*FA-GA)+1./GA-1./(2.*FA+GA)+1./GA)
 G264=(3.*A-1.)/4.*(-1./(2.*FA-GA)**2+1./(2.*FA+GA)**2)*DG
 G265=-DF/4.*(1./(FA+F-EA-GA)-1./(FA-F+EA-GA)-1./(FA+F-EA+GA)
1+1./(FA-F+EA+GA))
 G266=(2.*A-1.)/2.*DF*(1./(FA+F-EA-GA)**2+1./(FA-F+EA-GA)**2
1-1./(FA+F+GA-EA)**2-1./(FA-F+GA+EA)**2)
 G267=-DF/4.*(1./(FA+F-GA+EA)-1./(FA-F-EA-GA)-1./(GA+FA+F+EA)
1+1./(FA-F+GA-EA))
 G268=(2.*A-1.)/2.*DF*(1./(FA+F+EA-GA)**2+1./(FA-F-EA-GA)**2
1-1./(FA+F+EA+GA)**2-1./(FA-F+GA-EA)**2)
 G269=-DF/4.*(1./(FA+F-2.*GA)-1./(FA-F)-1./(FA+F)+1./(FA-F+2.*GA))
 G270=(2.*A-1.)/2.*DF*(1./(FA+F-2.*GA)**2+1./(FA-F)**2
1-1./(FA+F)**2-1./(FA-F+2.*GA)**2)
 G271=-DF/4.*(1./(F+FA)-1./(FA-F-2.*GA)-1./(F+FA+2.*GA)
1+1./(FA-F))
 G272=(2.*A-1.)/2.*DF*(1./(FA+F)**2+1./(FA-F-2.*GA)**2-1./(FA+F
1+2.*GA)**2-1./(FA-F)**2)
 G273=-DF/4.*(1./(2.*FA-EA-GA)+1./(GA-EA)-1./(2.*FA+GA-EA)+1.
1/(GA+EA))
 G279=(2.*A-1.)/2.*DF*(1./(2.*FA-EA-GA)**2+1./(GA-EA)**2
1-1./(2.*FA+GA-EA)**2-1./(GA+EA)**2)
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 G280=-DF/4.*(1./(2.*FA-GA+EA)+1./(EA+GA)-1./(2.*FA+GA+EA)
1+1./(GA-EA))
 G281=(2.*A-1.)/2.*DF*(1./(2.*FA+EA-GA)**2+1./(EA+GA)**2
1-1./(2.*FA+EA+GA)**2-1./(GA-EA)**2)
 G282=-DF/4.*(1./2./(FA-GA)-1./2./FA+1./2./GA)
 G283=(2.*A-1.)/2.*DF*(1./4./(2.*A-1.)**2+1./4./(FA-GA)**2
1-1./4./GA**2-1./4./FA**2)
 G284=-DF/4.*(1./2./FA+1./2./GA-1./2./(FA+GA) )
 G285=(2.*A-1.)/2.*(-DF)*(1./4./FA**2+1./4./GA**2-1./4./(FA+GA)**2
1-1./4./(2.*A-1.)**2)
 G285P=DG/4.*(1./(FA+F+GA)+1./(FA-F+GA)+1./(FA+F-GA)+1./(FA-F-GA))
 G286=(3.*A-1.)/4.*DG*(-2./GA**2+1./(2.*FA+GA)**2+1./(2.*FA-GA)**2)
 G287=DG/4.*(1./(2.*FA+GA)+1./(2.*FA-GA))
 G288=(2.*A-1.)/2.*DF*(1./(FA-F+GA+EA)**2-1./(F+FA-EA+GA)**2
1+1./(FA-F+EA-GA)**2-1./(F+FA-GA-EA)**2)
 G289=-DF/4.*(1./(FA+F+GA-EA)+1./(FA-F+GA-EA)+1./(FA+F-EA-GA)
1+1./(FA-F-GA+EA))
 G290=(2.*A-1.)/2.*DF*(1./(FA-F+GA-EA)**2-1./(FA+F+EA+GA)**2+
11./(FA-F-GA-EA)**2-1./(FA+F+EA-GA)**2)
 G291=-DF/4.*(1./(FA+F+EA+GA)+1./(FA-F+GA-EA)+1./(FA+F-GA+EA)
1+1./(FA-F-GA-EA))
 G292=(2.*A-1.)/2.*DF*(1./(FA-F+2.*GA)**2-1./(FA+F)**2+1.
1/(FA-F)**2-1./(F+FA-2.*GA)**2)
 G293=-DF/4.*(1./(F+FA)+1./(FA-F+2.*GA)+1./(FA+F-2.*GA)
1+1./(FA-F))
 G294=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(F+FA+2.*GA)**2
1+1./(FA-F-2.*GA)**2-1./(FA+F)**2)
 G295=-DF/4.*(1./(FA+F+2.*GA)+1./(FA-F)+1./(FA+F)+1./(FA-F-2.*GA))
 G296=(2.*A-1.)/2.*DF*(1./(EA+GA)**2-1./(2.*FA+GA-EA)**2
1+1./(EA-GA)**2-1./(2.*FA-GA-EA)**2)
 G297=-DF/4.*(1./(2.*FA+GA-EA)+1./(GA+EA)+1./(2.*FA-EA-GA)
1-1./(EA-GA))
 G298=(2.*A-1.)/2.*DF*(1./(EA-GA)**2-1./(2.*FA+EA+GA)**2
1+1./(EA+GA)**2-1./(2.*FA-GA+EA)**2)
 G299=-DF/4.*(1./(2.*FA+EA+GA)+1./(G A-EA)+1./(2.*FA-GA+EA)-1./(
1GA+EA))
 G300=(2.*A-1.)/2.*DF*(1./4./GA**2-1./4./FA**2+1./4./(2.*A-1.)
1**2-1./4./(FA-GA)**2)
 G301=-DF/4.*(1./2./FA+1./2./GA+1./2./(FA-GA))
 G302=(2.*A-1.)/2.*DF*(1./4./(2.*A-1.)**2-1./4./(FA+GA)**2+1./4.
1/GA**2-1./4./FA**2)
 G303=-DF/4.*(1./2./(FA+GA)+1./2./FA-1./2./GA)
 G304=-DF/2.*(1./(F+FA)-1./(F-FA))
 G305=(2.*A-1.)/2.*DF*(1./(F+FA-2.*EA)**2-1./(F+FA+2.*EA)**2
1-1./(F-FA-2.*EA)**2+1./(F-FA+2.*EA)**2)
 G306=  -DF/4.*(1./(F+FA+2.*EA)+1./(F+FA-2.*EA)+1./(FA-F+2.*EA)
1+1./(FA-F-2.*EA))
 G307=(2.*A-1.)/2.*DF*(1./(F+FA-GA+EA)**2-1./(F+FA+GA-EA)**2
1-1./(F-FA-GA+EA)**2+1./(F-FA+GA-EA)**2)
 G308=-DF/4.*(1./(F+FA+GA-EA)+1./(FA+F-GA+EA)-1./(F-FA+GA-EA)
1+1./(F-FA-GA+EA))
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 G309=(2.*A-1.)/2.*DF*(1./(F+FA-EA-GA)**2-1./(FA+F+GA+EA)**2
1-1./(F-GA-FA-EA)**2+1./(F-FA+GA+EA)**2)
 G310=-DF/4.*(1./(FA+F+GA+EA)+1./(FA+F-GA-EA)-1./(F-FA+EA+GA)
1-1./(F-FA-GA-EA))
 G311=(2.*A-1.)/2.*DF*(1./(FA+F-2.*GA)**2-1./(FA+F+2.*GA)**2
1-1./(F-FA-2.*GA)**2+1./(F-FA+2.*GA)**2)
 G312=-DF/4.*(1./(FA+F+2.*GA)+1./(F+FA-2.*GA)-1./(F-FA+2.*GA)
1-1./(F-FA-2.*GA))
 G313=-DF/2.*(1./(FA+F)+1./(FA-F))
 G314=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(FA+3.*F)**2+1./(
1(FA-3.*F)**2+4.*(2.*A-1.)**2)-1./(FA+F)**2)
 G315=-DF/4.*(1./(FA+3.*F)+1./(FA-F)+1./(FA+F)+(FA-3.*F)/
1((FA-3.*F)**2+4.*(2.*A-1.)**2))
 G316=(2.*A-1.)/2.* DF*(1./4./F**2-1./4./FA**2+1./4./(2.*A-1.)**2
1-1./4./(FA-F)**2)
 G317=-DF/4.*(1./2./FA+1./2./F+1./2./(FA-F))
 G318=(2.*A-1.)/2.*DF*(1./4./(2.*A-1.)**2-1./ 4./(FA+F)**2
1+1./4./F**2-1./4./FA**2)
 G319=-DF/4.*(1./2./(FA+F)+1./2./FA-1./2./F)
 G320=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(3.*FA+F)**2
1+1./(F+FA)**2-1./(3.*FA-F)**2)
 G321=-DF/4.*(1./(3.*FA+F)-1./(FA-F)+1./(3.*FA-F)-1./(FA+F))
 G322=(1.-EXP(2.*(A-1.)*C))/2.*(1./(F+FA)-1./(F-FA))
 G323=(1.-EXP(2.*A*C))/2.*(1./(F+FA)-1./(F-FA))
 G324=.5*C*EXP(2.*A*C)*(1./(F-FA)-1./(F+FA))+(EXP(2.*A*C)-1.)
1*2.*A*(1./(F+FA)**3-1./(F-FA)**3)
 G325=(1.-EXP((2.*A+SG)*C))/2.*(1./(F+FA)-1./(F-FA))
 G326=(3.*A-1.)/4.*DG*(1./(FA+F-EA)**2-1./(FA+F+EA)**2
1-1./(F-FA-EA)**2+1./(F-FA+EA)**2)
 G327=-DG/4.*(1./(F+FA+EA)+1./(F+FA-EA)-1./(F-FA+EA)-1./(F-FA-EA))
 G328=(3.*A-1.)/4.*DG*(1./(FA+F-GA)**2-1./(FA+F+GA)**2
1-1./(F-GA-FA)**2+1./(F-FA+GA)**2)
 G329=-DG/4.*(1./(FA+F+GA)+1./(F+FA-GA)-1./(F-FA+GA)-1./(F-FA-GA))
 G330=-DF/2.*(1./2./EA+1./2./(2.*FA-2.*EA)-1./4./(F+EA))
 G331=(2.*A-1.)*DF*(1./4./EA**2-1./8./(FA+EA)**2-1./8./(FA-EA)**2)
 G332=-DF/2.*(1./(GA-EA)+1./2./(2.*FA-GA+EA)-1./2./(2.*FA+GA-EA))
 G333=(2.*A-1.)*DF*(1./(GA-EA)**2-1./2./(2.*FA-GA+EA)**2-1./2.
1/(2.*FA+GA-EA)**2)
 G334=-DF/2.*(1./2./(EA+GA)+1./2./(2.*FA-EA-GA)-1./2./(2.*FA+EA+
1GA))
 G335=(2.*A-1.)*DF*(1./(EA+GA)**2-1./2./(2.*FA+EA+GA)**2
1-1./2./(2.*FA-EA-GA)**2)
 G336=-DF/2.*(1./2./GA+1./2./(2.*FA-2.*GA)-1./4./(FA+GA))
 G337=(2.*A-1.)*DF*(1./4./GA**2-1./8./(FA+GA)**2-1./8./(FA-GA)**2)
 G338=-DF/4.*(1./F-1./2./(F+FA)+1./2./(FA-F))
 G339=(2.*A-1.)/2.*DF*(1./2./F**2-1./4./(FA+F)**2-1./4./(FA-F)**2)
 G340=-DF/4.*(2./(FA-F)-1./(3.*FA-F)+1./(FA+F))
 G341=(2.*A-1.)/2.*DF*(2./(FA-F)**2-1./(3.*FA-F)**2-1./(FA+F)**2)
 G342=-DF/4.*(1./(FA+F)-1./(3.*FA+F)+1./(FA-F))
 G343=(2.*A-1.)/2.*DF*(1./(FA+F)**2-1./(3.*FA+F)**2-1./(FA-F)**2)
 G344=-DF/4.*(1./FA-1./4./FA)
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 G345=(2.*A-1.)/2.*DF*(1./2./FA**2-1./16./FA**2-1./4./(2.*A-1.)**2)
 G346=EXP(2.*A*C)/2.*(C/2./A-1./4./A**2)+1./8./A**2-A*C*EXP(2.*A*C)
1/4./FA**2-1./8./FA**2*(EXP(2.*A*C)-1.)
 G347=DG/4.*(2./EA+1./(2.*FA-EA)-1./(2.*FA+EA))
 G348=(3.*A-1.)/4.*DG*(1./(2.*FA+EA)**2+1./(2.*FA-EA)**2-2./EA**2)
 G349=DG/4.*(2./GA+1./(2.*FA-GA)-1./(2.*FA+GA))
 G350=(3.*A-1.)/4.*DG*(1./(2.*FA+GA)**2+1./(2.*FA-GA)**2-2./GA**2)
 G351=-DF/4./FA
 G352=(2.*A-1.)/2.*DF*(1./(2.*FA-2.*EA)**2-1./4./(FA+EA)**2)
 G353=-DF/4.*(1./2./(FA+EA)+1./2./(FA-EA))
 G354=(2.*A-1.)/2.*DF*(1./(2.*FA-GA+EA)**2-1./(2.*FA+GA-EA)**2)
 G355=-DF/4.*(1./(2.*FA+GA-EA)+1./(2.*FA-GA+EA))
 G356=(2.*A-1.)/2.*DF*(1./(2.*FA-GA-EA)**2-1./(2.*FA+GA+EA)**2)
 G357=-DF/4.*(1./(2.*FA+EA+GA)+1./(2.*FA-EA-GA))
 G358=(2.*A-1.)/2.*DF*(1./2./(FA-GA)**2-1./2./(FA+GA)**2)
 G359=-DF/4.*(1./2./(FA+GA)+1./2./(FA-GA))
 G360=(2.*A-1.)/2.*DF*(1./4./(FA-F)**2-1./4./(FA+F)**2)
 G361=-DF/4.*(1./2./(FA+F)+1./2./(FA-F))
 G362=(2.*A-1.)/2.*DF*(1./(FA+F)**2-1./(3.*FA-F)**2)
 G363=-DF/4.*(1./(3.*FA-F)+1./(FA+F))
 G364=(2.*A-1.)/2.*DF*(1./(FA-F)**2-1./(3.*FA+F)**2)
 G365=-DF/4.*(1./(3.*FA+F)+1./(FA-F))
 G366=-DF/16./FA
 G367=(1.-EXP(2.*(A-1.)*C))/4./FA
 G368=(1.-EXP(2.*A*C))/4./FA
 G369=-C*EXP(2.*A*C)/4./FA+A/8./FA**3*(EXP(2.*A*C)-1.)
 G370=(1.-EXP((2.*A+SG)*C))/4./FA
 G371=(3.*A-1.)/4.*DG*(-1./(2.*FA-EA)**2+1./(2.*FA+EA)**2)
 G372=DG/4.*(1./(2.*FA+EA)+1./(2.*FA-EA))
 G373=(3.*A-1.)/4.*DG*(-1./(2.*FA-GA)**2+1./(2.*FA+GA)**2)
 G374=DG/4.*(1./(2.*FA+GA)+1./(2.*FA-GA))
 R1=EB*(E1*G1+E2*G2)-D/E*(E5*G7+E6*G8)
 R2=EB*(E3*G3+E4*G4)-D/E*(E7*G9+E8*G10)
 R3A=-D/E*(E9*G11+E10*G12+E11*G13+E12*G14+E13*G15+E14*G16+E15*G17
1+E16*G18)-A1H*A*(E5*G45+E6*G45P)-A1H*EA*(E5*G64+E6*G65)-A2H*A
2*(E5*G84+E6*G85)-A2H*GA*(E5*G104+E6*G105)-A1H*B100*(E68*G45
3+E69*G45P)-A1H*B101*(E68*G64+E69*G65)-A2H*B102*(E68*G84+E69*G85
4)-A2H*B103*(E68*G104+E69*G105)
 R4A=-D/E*(E17*G19+E18*G20+E19*G21+E20*G22+E21*G23+E22*G24+E23*G25
1+E24*G26)-A1H*A*(E7*G46+E8*G47)-A1H*EA*(E7*G66+E8*G67)-A2H*A*(E7*
2G86+E8*G87)-A2H*GA*(E7*G106+E8*G107)-A1H*B100*(E70*G46+E71*G47
3)-A1H*B101*(E70*G66+E71*G67)-A2H*B102*(E70*G86+E71*G87)-A2H*B103
4*(E70*G106+E71*G107)
 R5A=-A1H*A*(E9*G48+E10*G49+E11*G50+E12*G51+E13*G52+E14*G53+E15
1*G54+E16*G55)-A1H*EA*(E9*G68+E10*G69+E11*G70+E12*G71+E13*G72
2+E14*G73+E15*G74+E16*G75)-A2H*A*(E9*G88+E10*G89+E11*G90+E12*
3G91+E13*G92+E14*G93+E15*G94+E16*G95)-A2H*GA*(E9*G108+E10*G109+E11
4*G110+E12*G111+E13*G112+E14*G113+E15*G114+E16*G115)-A1H*B100*(E72
5*G48+E73*G49+E74*G50+E75*G51+E76*G52+E77*G53+E78*G54+E79*G55)
6-A1H*B101*(E72*G68+E73*G69+E74*G70+E75*G71+E76*G72+E77*G73+E78*
7G74+E79*G75)-A2H*B102*(E72*G88+E73*G89+E74*G90+E75*G91+E76*G92
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8+E77*G93+E78*G94+E79*G95)-A2H*B103*(E72*G108+E73*G109+E74*G110
9+E75*G111+E76*G112+E77*G113+E78*G114+E79*G115)
 R6A=-A1H*A*(E17*G56+E18*G57+E19*G58+E20*G59+E21*G60+E22*G61+E23
1*G62+E24*G63)-A1H*EA*(E17*G76+E18*G77+E19*G78+E20*G79+E21*G80
2+E22*G81+E23*G82+E24*G83)-A2H*A*(E17*G96+E18*G97+E19*G98+E20*G99
3+E21*G100+E22*G101+E23*G102+E24*G103)-A2H*GA*(E17*G116+E18*G117+
4E19*G118+E20*G119+E21*G120+E22*G121+E23*G122+E24*G123)-A1H*B100
5*(E80*G56+E81*G57+E82*G58+E83*G59+E84*G60+E85*G61+E86*G62+E87*
6G63)-A1H*B101*(E80*G76+E81*G77+E82*G78+E83*G79+E84*G80+E85*G81
7+E86*G82+E87*G83)-A2H*B102*(E80*G96+E81*G97+E82*G98+E83*G99+E84
8*G100+E85*G101+E86*G102+E87*G103)-A2H* B103*(E80*G116+E81*G117
9+E82*G118+E83*G119+E84*G120+E85*G121+E86*G122+E87*G123)
 R5B=G*G*(E25*G124+E26*G125+E27*G126+E28*G127+E29*G128+E30*G129
1+E31*G130+E32*G131+E33*G132)-F*(E25*G149+E26*G150+E27*G151+E28*
2G152+E29*G153+E30*G154+E31*G155+E32*G156+E33*G157)-B104*(E87P
3*G124+E88*G125+E89*G126+E90*G127+E91*G128+E92*G129+E93*G130+E94
4*G131+E95*G132)-B105*(E87P*G149+E88*G150+E89*G151+E90*G152+E91*
5 G153+E92*G154+E93*G155+E94*G156+E95*G157)
 R6B=G*G*(E25*G174+E26*G175+E27*G176+E28*G177+E29*G178+E30*G179
1+E31*G180+E32*G181+E33*G182)-FA*(E25*G199+E26*G200+E27*G201
2+E28*G202+E29*G203+E30*G204+E31*G205+E32*G206+E33*G207)-B106*
3(E87P*G174+E88*G175+E89*G176+E90*G177+E91*G178+E92*G179+E93*
4G180+E94*G181+E95*G182)-B107*(E87P*G199+E88*G200+E89*G201+E90*
5G202+E91*G203+E92*G204+E93*G205+E94*G206+E95*G207)
 R7A=G*G*(E34*G124+E35*G133+E36*G134)+G*G/2.*(
1E44*G124+E45*G133+E46*G134)+G*G*(-D53*G141+2.*D63*G142+3.
2*D58*G142+2.*D58*G143)-F*(E34*G149+E35*G158+E36*G159)-F/2.*(E44
3*G149+E45*G158+E46*G159)-F*(-D53*G166+2.*D63*G167+3.*D58*G167
4+2.*D58*G168)-B104*(E96*G124+E97*G133+E98*G134)-.5*B104*(E99*
5G124+E100*G133+E101*G134)-B104*(D53*G141+2.*D63*G142+D58*G142+
62.*D58*G143)-B105*(E96*G149+E97*G158+E98*G159)-B105/2.*(E99*
7G149+E100*G158+E101*G159)-B105*(D53*G166+2.*D63*G167+D58*
8G167+D58*G168)
 R8AP=G*G*(E37*G135+E38*G136+E39*G137+E40*G138)+G*G/2.*(E47*G135
1+E48*G136+E49*G137+E50*G138)+G*G*(-D54*G141+2.*D64*G142+3.*D59
2*G142+2.*D59*G143)-F*(E37*G160+E38*G161+E39*G162+E40*G163)-F/2.
3*(E47*G160+E48*G161+E49*G162+E50*G163)-F*(-D54*G166+2.*D64*G167
4+3.*D59*G167+2.*D59*G168)+G*G*(E34*G174+E35*G183+E36*G184)+G*G/2.
5*(E44*G174+E45*G183+E46*G184)+G*G*(-D53*G191+2.*D63*G192+3.*D58
6*G192+2.*D58*G193)-FA*(E34*G199+E35*G208+E36*G209)-FA/2.*(E44
7*G199+E45*G208+E46*G209)-FA*(-D53*G216+2.*D63*G217
8+3.*D58*G217+2.*D58*G218)-B104*(E102*G135+E103*G136+E104*G137+E105
9*G138)-B104/2.*(E106*G135+E107*G136+E108*G137+E109*G138)
 R8ADP=-B104*(D54*G141+D64*G142+D59*G142+2.*D59*G143)-B105*(E102
1*G160+E103*G161+E104*G162+E105*G163)-B105/2.*(E106*G160+E107*
2G161+E108*G162+E109*G163)-B105*(D54*G166+2.*D64*G167+D59*G167+2.*
3D59*G168)-B106*(E96*G174+E97*G183+E98*G184)-B106/2.*(E99*G174
4+E100*G183+E101*G184)-B106*(D53*G191+2.*D63*G192+D58*G192+2.*D58*
5G193)-B107*(E96*G199+E97*G208+E98*G209)-B107/2.*(E99*G199+E100
6*G208+E101*G209)-B107*(D53*G216+2.*D63*G217+D58*G217+2.*D58*G218)
 R9AP=G*G*(E41*G124+E42*G139+E43*G140)+G*G/2.*(E50P*G124+E51*G139
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1+E52*G140)+G*G*(-D55*G141+2.*G142*D65+3.*D60*G142+2.*D60*G143)
2-F*(E41*G149+E42*G164+E43*G165)-F/2.*(E50P*G149+E51*G164+E52*G165)
3-F*(-D55*G166+2.*D65*G167+3.*D60*G167+2.*D60*G168)+G*G*(E37*G185
4+E38*G186+E39*G187+E40*G188)+G*G/2.*(E47*G185+E48*G186+E49*G187+
5E50*G188)+G*G*(-D54*G191+2.*D64*G192+3.*D59*G192+2.*D59*G193)
6-FA/2.*(E47*G210+E48*G211+E49*G212+E50*G213)-FA*(-D54*G216+2.*D64
7*G217+3.*D59*G217+2.*D59*G218)-B104*(E110*G124+E111*G139+E112*
8G140)-B104/2.*(E113*G124+E114*G139+E115*G140)-B104*(D55*G141+2. *
9D65*G142+D60*G142+2.*D60*G143)
 R9ADP=-B105/2.*(E113*G149+E114*G164+E115*G165)-B105*(D55*G166+
12.*D65*G167+D60*G167+2.*D60*G168)-B106*(E102*G185+E103*G186+
2E104*G187+E105*G188)-B106/2.*(E106*G185+E107*G186+E108*G187+E109
3*G188)-B106*(D54*G191+2.*D64*G192+D59*G192+2.*D59*G193)-B107*
4(E102*G210+E103*G211+E104*G212+E105*G213)-B107/2.*(E106*G210
5+E107*G211+E108*G212+E109*G213)-B107*(D54*G216+2.*D64*G217+D59
6*G217+2.*D59*G218)-FA*(E37*G210+E38*G211+E39*G212+E40*G213)
8-B105*(E110*G149+E111*G164+E112*G165)
 R10A=G*G*(E41*G174+E42*G189+E43*G190)+G*G/2.*(E50P*G174+E51*
1G189+E52*G190)+G*G*(-D55*G191+2.*D65*G192+3.*D60*G192+2.*D60*G193)
2-FA*(E41*G199+E42*G214+E43*G215)-FA/2.*(E50P*G199+E51*G214+
3E52*G215)-FA*(-D55*G216+2.*D65*G217+3.*D60*G217+2.*D60*G218)-B106
4*(E110*G174+E111*G189+E112*G190)-B106/2.*(E113*G174+E114*G189
5+E115*G190)-B106*(D55*G191+2.*D65*G192+D60*G192+2.*D60*G193)-B107
6*(E110*G199+E111*G214+E112*G215)-B107/2.*(E113*G199+E114*G214
7+E115*G215)-B107*(D55*G216+2.*D65*G217+D60*G217+2.*D60*G218)
 R7B=(1.-A)*G*(E58*G124+E59*G133+E60*G134)-F*(E58*G149+E59*G158
1+E60*G159)
 R8B=(1.-A)*G*(E61*G135+E62*G136+E63*G137+E64*G138)-F*(E61*G160+
1E62*G161+E63*G162+E64*G163)+(1.-A)*G*(E58*G174+E59*G183+E60*G184)
2-FA*(E58*G199+E59*G208+E60*G209)
 R9B=(1.-A)*G*(E65*G124+E66*G139+E67*G140)-F*(E65*G149+E66*G164+
1E67*G165)+(1.-A)*G*(E61*G185+E62*G186+E63*G187+E64*G188)-FA*(
2E61*G210+E62*G211+E63*G212+E64*G213)
 R3BP=RATIO*G*G*(-D57*G141+D67*B142*2.+3.*D62*G142+2.*D62*G143)
1+G*G*E53*G144+G*G*D/E*(E54*G145+E55*G146+E56*G147+E57*G148)
2+G*G*(-A1P*G141+2.*A2P*G142+3.*A3P*G142+2.*A3P*G143)-RATIO*F*(
3-D57*G166+2.*D67*G167+3.*D62*G167+2.*D62*G168)-F*E53*G169
4-F* D/E*(E54*G170+E55*G171+E56*G172+E57*G173)-F*(-A1P*G166+2.*
5A2P*G167+3.*A3P*G167+2.*A3P*G168)-RATIO*B104*(D57*G141+2.*D67*G142
6+D62*G142+2.*D62*G143)-B104*E116*G144-B104*D/E*(E117*G145
7+E118*G146+E119*G147+E120*G148)-B104*(A1P*G141+2.*A2P*G142+A3P
8*G142+2.*A3P*G143)-B105*RATIO*(D57*G166+2.*D67*G167+D62*G167+2.*
9D67*G168)-B105*E116*G169
 R10B=(1.-A)*G*(E65*G174+E66*G189+E67*G190)-FA*(E65*G199+E66
1*G214+E67*G215)
 R3BDP=-B105*D/E*(E117*G170+E118*G171+E119*G172+E120*G173)-
1B105*(A1P*G166+2.*A2P*G167+A3P*G167+2.*A3P*G168)
 R4BP=RATIO*G*G*(-D57*G191+2.*D67*G192+3.*D62*G192+2.*D62*G193)+
1G*G*E53*G194+G*G*(E54*G195+E55*G196+E56*G197+E57*G198)*D/E
2+G*G*(-A1P*G191+2.*A2P*G192+3.*A3P*G192+2.*A3P*G193)-FA*RATIO*(
3-D57*G216+2.*D67*G217+3.*D62*G217+2.*D62*G218)-FA*E53*G219-FA
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4*D/E*(E54*G220+E55*G221+E56*G222+E57*G223)-FA*(-A1P*G216+2.*A2P
5*G217+3.*A3P*G217+2.*A3P*G218)-RATIO*B106*(D57*G191+2.*D67*G192
6+D62*G192+2.*D62*G193)-B106*E116*G194-B106*D/E*(E117*G195
7+E118*G196+E119*G197+E120*G198)-B106*(A1P*G191+2.*A2P*G192+A3P
8*G192+2.*A3P*G193)-RATIO*B107*(D57*G216+2.*D67*G217+D62*G217+2.
9*D62*G218)-B107*E116*G219
 R4BDP=-B107*D/E*(E117*G220+E118*G221+E119*G222+E120*G223)-B107*(
1A1P*G216+2.*A2P*G217+A3P*G217+2.*A3P*G218)
 R5C=G*G*(-D56*G141+2.*D66*G142+3.*D61*G142+2.*D61*G143)-F*(-D56
1*G166+2.*D66*G167+3.*D61*G167+2.*D61*G168)-B104*(D56*G141+2.*D66
2*G142+D61*G142+2.*D61*G143)-B105*(D56*G166+2.*D66*G167+D61*
3G167+2.*D61*G168)
 R6C=G*G*(-D56*G191+2.*D66*G192+3.*D61*G192+2.*D61*G193)-FA*(-D56
1*G216+2.*D66*G217+3.*D61*G217+2.*D61*G218)-B106*(D56*G191+2.*
2D66*G192+D61*G192+2.*D61*G193)
3-B107*(D56*G216+2.*D66*G217+D61*G217+2.*D61*G218)
 R8A=R8AP+R8ADP
 R9A=R9AP+R9ADP
 R3B=R3BP+R3BDP
 R4B=R48P+R4BDP
 R3=R3A+R3B
 R4=R4A+R4B
 R5=R5A+R5B+R5C
 R6=R6A+R6B+R6C
 R7=R7A+R7B
 R8=R8A+R8B
 R9=R9A+R9B
 R10=R10A+R10B
 S1=EB*(E1*G3+E2*G5)-D/E*(E5*G9+E6*G29)
 S2=EB*(E3*G1 +E4*G6)-D/E*(E7*G7+E8*G48)
 S3A=-D/E*(E9*G29+E10*G30+E11*G31+E12*G32+E13*G33+E14*G34+E15*
1G35+E16*G36)-A1H*A*(E5*G195+E6*G224)-A1H*EA*(E5*G196+E6*G243)
2-A2H*A*(E5*G197+E6*G262)-A2H* GA*(E5*G198+E6*G285P)-A1H*B100*(E68
3*G195+E69*G224)-A1H*B101*(E68*G196+E69*G243)-A2H* B102*(E68*G197
4+E69*G262)-A2H*B103*(E68*G198+E69*G285P)
 S4A=-D/E*(E17*G37+E18*G38+E19*G39+E20*G40+E21*G41+E22*G42+E23
1*G43+E24*G44)-A1H*A*(E7*G225+E8*G226)-A1H*EA*(E7*G244+E8*G245)
2-A2H*A*(E7*G263+E8*G264)-A2H*GA*(E7*G286+E8*G287)-A1H*B100*(E70
3*G225+E71*G226)-A1H*B101*(E70*G244+E71*G245)-A2H*B102*(E70*G263
4+E71*G264)-A2H*B103*(E70*G286+E71*G287)
 S5AP=-A1H*A*(E9*G227+E10*G228+E11*G229+E12*G230+E13*G231
1+E14*G232+E15*G233+E16*G234)-A1H*EA*(E9*G246+E10*G247+E11*G248
2+E12*G249+E13*G250+E14*G251+E15*G252+E16*G253)-A2H*A*(E9*G265
3+E10*G266+E11*G267+E12*G268+E13*G269+E14*G270+E15*G271+E16*G272)
4-A2H*GA*(E9*G288+E10*G289+E11*G290+E12*G291+E13*G292+E14*G293
5+E15*G294+E16*G295)-A1H*B100*(E72*G227+E73*G228+E74*G229
6+E75*G230+E76*G231+E77*G232+E78*G233+E79*G234)-A1H*B101*(E72*
7G246+E73*G247+E74*G248+E75*G249+E76*G250+E77*G251+E78*G252+E79*
8G253)-A2H*B102*(E72*G265+E73*G266+E74*G267+E75*G268+E76*G269+E77*
9G270+E78*G271+E79*G272)-A2H*B103*(E72*G288+E73*G289+E74*G290)
 S5ADP=-A2H*B103*(E75*G291+E76*G292+E77*G293+E78*G294+E79*G295)
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 S6AP=-A1H*A*(E17*G235+E18*G236+E19*G237+E20*G238+E21*G239+E22
1*G240+E23*G241+E24*G242)-A1H*EA*(E17*G254+E18*G255+E19*G256+E20
2*G257+E21*G258+E22*G259+E23*G260+E24*G261)-A2H*A*(E17*G273+E18
3*G279+E19*G280+E20*G281+E21*G282+E22*G283+E23*G284+E24*G285)
4-A2H*GA*(E17*G296+E18*G297+E19*G298+E20*G299+E21*G300+E22*G301+
5E23*G302+E24*G303)-A1H*B100*(E80*G235+E81*G236+E82*G237+E83*G238
6 +E84*G239+E85*G240+E86*G241+E87*G242)-A1H*B101*(E80*G254+E81
7*G255+E82*G256+E83*G257+E84*G258+E85*G259+E86*G260
8+E87*G261)-A2H*B102*(E80*G273+E81*G279+E82*G280+E83*G281+E84
9*G282+E85*G283+E86*G284+E87*G285)
 S6ADP=-A2H*B103*(E80*G296+E81*G297+E82*G298+E83*G299+E84*G300+
1E85*G301+E86*G302+E87*G303)
 S5B=G*G*(E25*G174+E26*G175+E27*G176+E28*G177+E29*G178+E30*G179
1+E31*G180+E32*G181+E33*G182)-F*(E25*G304+E26*G305+E27*G306+E28*
2G307+E29*G308+E30*G309+E31*G310+E32*G311+E33*G312)-B104*(E87P*G174
3+E88*G175+E89*G176+E90*G177+E91*G178+E92*G179+E93*G180+E94
4*G181+E95*G182)-B105*(E87P*G304+E88*G305+E89*G306+E90*G307+E91
5*G308+E92*G309+E93*G310+E94*G311+E95*G312)
 S6B=G*G*(E25*G124+E26*G330+E27*G331+E28*G332+E29*G333+E30*G334
1+E31*G335+E32*G336+E33*G337)-FA*(E25*G351+E26*G352+E27
2*G353+E28*G354+E29*G355+E30*G356+E31*G357+E32*G358+E33*G359)
3-B106*(E87P*G124+E88*G330+E89*G331+E90*G332+E91*G333+E92*G334
4+E93*G335+E94*G336+E95*G337)-B107*(E87P*G351+E88*G352+E89*G353
5+E90*G354+E91*G355+E92*G356+E93*G357+E94*G358+E95*G359)
 S7A=G*G*(E34*G174+E35*G183+E36*G184)+G*G/2.*(E44*G174+E45*G183+
1E46*G184)+G*G*(-D53*G191+2.*D63*G192+3.*D58*G192+2.*D58*G193)
2-F*(E34*G313+E35*G314+E36*G315)-F/2.*(E44*G313+E45*G314+E46*G315)
3-F*(-D53*G322+2.*D63*G323+3.*D58*G323+2.*D58*G324)-B104*(E96*G174
4+E97*G183+E98*G184)-B104/2.*(E99*G174+E100*G183+E101*G184)-B104*
5(D53*G191+2.*D63*G192+D58*G192+2.*G193*D58)-B105*(E96*G313
6+E97*G314+E98*G315)-B105/2.*(E99*G313+E100*G314+E101*G315)-B105
7*(D53*G322+2.*D63*G323+D58*G323+2.*D58*G324)
 S8AP=G*G*(E37*G185+E38*G186+E39*G187+E40*G188)+G*G/2.*(E47*G185+
1E48*G186+E49*G187+E50*G188)+G*G*(-D54*G191+2.*D64*G192+3.*D59*
2G192+2.*D59*G193)-F*(E37*G316+E38*G317+E39*G318+E40*G319)-F
3*(-D54*G322+2.*D64*G323+3.*D59*G323+2.*D59*G324)+G*G*(E34*G124+
4E35*G338+E36*G339)+G*G*(E44*G124+E45*G338+E46*G339)+G*G*(-D53*G141
5+2.*D63*G142+3.*D58*G142+2.*D58*G143)-FA*(E34*G351+E35*G360+E36
6*G361)-FA/2.*(E44*G351+E45*G360+E46*G361)-FA*(-D53*G367+2.*D63
7*G368+3.*D58*G368+2.*D58*G369)-B104*(E102*G185+E103*G186+E104
8*G187+E105*G188)-B104/2.*(E106*G185+E107*G186+E108*G187+E109*G188
9)-B104*(D54*G191+2.*D64*G192+D59*G192+2.*D59*G193)
 S8ADP=-B105*(E102*G316+E103*G317+E104*G318+E105*G319)-B105/2.*(
1E106*G316+E107*G317+E108*G318+E109*G319)-B105*(D54*G322+2.*
2D64*G323+D59*G323+2.*D59*G324)-B106*(E96*G124+E97*G338+E98*G339)
3-B106/2.*(E99*G124+E100*G338+E101*G339)-B106*(D53*G141+2.*D63*
4G142+D58*G142+2.*D58*G143)-B107*(E96*G351+E97*G360+E98*G361)-
5B107/2.*(E99*G351+E100*G360+E101*G361)-B107*(D53*G367+2.*D63
6*G368+D58*G368+2.*D58*G369)-F/2.*(E47*G316+E48*G317+E49*G318+E50*
7G319)
 S9AP=G*G*(E41*G174+E42*G189+E43*G190)+G*G/2.*(E50P*G174+E51*G189
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1+E52*G190)+G*G*(-D55*G191+2.*D65*G192+3.*D60*G192+2.*D60*G193)-F*(
2E41*G313+E42*G320+E43*G321)-F/2.*(E50P*G313+E51*G320+E52*G321)-F*(
3-D55*G322+2.*D65*G323+3.*D60*G323+2.*D60*G324)+G*G*(E37*G340
4+E38*G341+E39*G342+E40*G343)+G*G/2.*(E47*G340+E48*G341+E49*G342+
5E50*G343)+G*G*(-D54*G141+2.*D64*G142+3.*D59*G143)-FA*(E37*G362
6+E38*G363+E39*G364+E40*G365)-FA/2.*(E47*G362+E48*G363+E49*G364
7+E50*G365)-FA*(-D54*G367+2.*D64*G368+3.*D59*G368+2.*D59*G369)
8-B104*(E110*G174+E111*G189+E112*G190)-B104/2.*(E113*G174+E114*G189
9+E115*G190)+G*G*2.*D59*G143
 S9ADP=-B104*(D55*G191+2.*D65*G192+D60*G192+2.*D60*G193)-B105*(
1E110*G313+E111*G320+E112*G321)-B105/2.*(E113*G313+E114*G320+
2E115*G321)-B105*(D55*G322+2.*D65*G323+D60*G323+2.*D60*G324)-B106
3*(E102*G340+E103*G341+E104*G342+E105*G343)-B106/2.*(E106*G340
4+E107*G341+E108*G342+E109*G343)-B106*(D54*G141+2.*D64*G142+2.*
5D59*G143+D59*G142)-B107*(E102*G362+E103*G363+E104*G364+E105*G365)
6-B107/2.*(E106*G362+E107*G363+E108*G364+E109*G365)-B107*(D54
7*G367+2.*D64*G368+D59*G368+2.*D59*G369)
 S10A=G*G*(E41*G124+E42*G344+E43*G345)+G*G/2.*(E50P*G124+E51*G344
1+E52*G345)+G*G*(-D55*G141+2.*D65*G142+3.*D60*G142+2.*D60*G143)
2-FA*(E41*G351+E42*G158+E43*G366)-FA/2.*(E50P*G351+E51*G158+
4E52*G366)-FA*(-D55*G367+2.*D65*G368+3.*D60*G368+2.*D60*G369)-
58106*(E110*G124+E111*G344+E112*G345)-B106/2.*(E113*G124+E114*G344
6+E115*G345)-B106*(D55*G141+2.*D65*G142+D60*G142+2.*D60*G143)-B107
7*(E110*G351+E111*G158+E112*G366)-B107/2.*(E113*G351+E114*G158+
8E115*G366)-B107*(D55*G367+2.*D65*G368+D60*G368+2.*D60*G369)
 S7B=(1.-A)*G*(E58*G174+E59*G183+E60*G184)-F*(E58*G313+E59*G314
1+E60*G315)
 S8B=(1.-A)*G*(E61*G185+E62*G186+E63*G187+E64*G188)-F*(E61*G316
1+E62*G317+E63*G318+E64*G319)+(1.-A)*G*(E58*G124+E59*G338+E60*
2G339)-FA*(E58*G351+E59*G360+E60*G361)
 S9B=(1.-A)*G*(E65*G174+E66*G189+E67*G190)-F*(E65*G313+E66*G320
1+E67*G321)+(1.-A)*G*(E61*G340+E62*G341+E63*G342+E64*G343)-FA*(E61
2*G362+E62*G363+E63*G364+E64*G365)
 S10B=(1.-A)*G*(E65*G124+E66*G344+E67*G345)-FA*(E65*G351+E66*G158+
1E67*G366)
 S38P=RATIO*G*G*(-D57*G191+2.*D67*G192+3.*D62*G192+2.*D62*G193)
1+G*G*E53*G194+G*G*(E54*G195+E55*G196+E56*G197+E57*G198)*D/E
2+G*G*(-A1P*G191+2.*A2P*G192+3.*A3P*G192+2.*A3P*G193)-RATIO*F *(
3-D57*G322+2.*D67*G323+3.*D62*G323+2.*D62*G324)-F*E53*G325
4-F*D/E*(E54*G326+E55*G327+E56*G328+E57*G329)-F*(-A1P*G322+2.*A2P
5*G323+3.*A3P*G323+2.*A3P*G324)-B104*RATIO*(D57*G191+2.*D67*G192
6+D62*G192+2.*D62*G193)-B104*E116*G194-B104*D/E*(E117*G195
7+E118*G196+E119*G197+E120*G198)-B104*(A1P*G191+2.*A2P*G192+A3P*
8G192+2.*A3P*G193)-B105*RATIO*(D57*G322+2.*D67*G323+D62*G323+
92.*D62*G324)-B105*E116*G325
 S38DP=-B105*D/E*(E117*G326+E118*G327+E119*G328+E120*G329)-B105*
1(A1P*G322+2.*A2P*G323+A3P*G323+2.*A3P*G324)
 S4BP=G*G*RATIO*(-D57*G141+2.*D67*G142+3.*D62*G142+2.*D62*G143)
1+G*G*E53*G144+G*G*D/E*(E54*G347+E55*G348+E56*G349+E57*G350)
2+G*G*(-A1P*G141+2.*A2P*G142+3.*A3P*G142+2.*A3P*G346)-FA*RATIO*
3(-D57*G367+2.*D67*G368+3.*D62*G368+2.*D62*G369)-E53*G370*FA
```

```
4-FA*D/E*(E54*G371+E55*G372+E56*G373+E57*G374)-FA*(-A1P*G367+2.
5*A2P*G368+3.*A3P*G368+2.*A3P*G369)-RATIO*B106*(D57*G141+2.*D67*
6G142+D62*G142+2.*D62*G346)-B106*E116*G144-B106*D/E*(E117*G347
7 +E118*G348+E119*G349+E120*G350)-B106*(A1P*G141+2.*A2P*G142+A3P
8*G142+2.*A3P*G346)-B107*RATIO*(D57*G367+2.*D67*G368+D62*G368
9+2.*D62*G369)-E116*G370*B107
 S4BDP=-B107*D/E*(E117*G371+E118*G372+E119*G373+E120*G374)-B107*(
1A1P*G367+2.*A2P*G368+A3P*G368+2.*A3P*G369)
 S5C=G*G*(-D56*G191+2.*D66*G192+3.*D61*G192+2.*D61*G193)-F*(-D56
1*G322+2.*D66*G323+3.*D61*G323+2.*D61*G324)-B104*(D56*G191+2.*
2D66*G192+D61*G192+2.*D61*G193)-B105*(D56*G322+2.*D66*G323+D61
3*G323+2.*D61*G324)
 S6C=G*G*(-D56*G141+2.*D66*G142+3.*D61*G142+2.*D61*G143)-FA*(
1-D56*G367+2.*D66*G368+3.*D61*G368+2.*D61*G369)-B106*(D56*G141
2+2.*D66*G142+D61*G142+2.*D61*G143)-B107*(D56*G367+2.*D66*G368+
3D61*G368+2.*D61*G369)
 S5A=S5AP+S5ADP
 S6A=S6AP+S6ADP
 S3B=S3BP+S3BDP
 S4B=S4BP+S4BDP
 S3=S3A+S3B
 S4=S4A+S4B
 S5=S5A+S5B+S5C
 S6=S6A+S6B+S6C
 S7=S7A+S7B
 S8=S8A+S8B
 S9=S9A+S9B
 S10=S10A+S10B
 U1=PHI**3*R7+PHI**2*R8+PHI*R9+R10
 U2=-(R1*PHI+R2)/U1
 U3=-(R3*PHI+R4)/U1
 U4=-(R5*PHI+R6)/U1
 U5=PHI**3*S7+PHI**2*S8+PHI*S9+S10
 U6=S5*PHI+S6+U5*U4
 U7=S3*PHI+S4+U5*U3
 U8=S1*PHI+S2+U5*U2
 U9=U7**2-4.*U6*U8
 AT01=(-U7+U9**.5)/2./U6/D**2
 AT02=(-U7-U9**.5)/2./U6/D**2
 WRITE(6,8) BETA,D,X,Q,RATIO,SG,C,PHI,U9,AT01,AT02,U1,U2,U3,U4,
1U5,U6,U7,U8,R1,R2,R3,R4,R5,R6,R7,R8,R9,R10,S1,S2,S3,S4,S5,S6,S7,
2S8,S9,S10,A1H,A2H,A1P,A2P,A3P
8 FORMAT (8E12.3)
 GO TO 1
 END
```

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BIOGRAPHICAL SKETCH

The author was born on February 1, 1938, in Tientsin, China. He received his Bachelor of Science degree in Mechanical Engineering in July, 1959, from the National Taiwan University, Taipei, Taiwan, China. In October, 1959, he began two years of service in the Air Force of the Republic of China. Upon release from the service, he enrolled in the Graduate School of the University of Florida in September, 1962, and received the degree of Master of Science with a major in Engineering Science and Mechanics in April, 1964. Thereafter, he continued his studies and research leading to the degree of Doctor of Philosophy in the Department of Engineering Science and Mechanics.

He is a member of Phi Kappa Phi.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June, 1967

M. G. Forsman

Dean, College of Engineering

Dean, Graduate School

Supervisory Committee:

S. J. Lu

Chairman

W. A. Nash

J. R. Haigh

R. M. Blake

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